Kempf–Ness type theorems and Nahm's equations

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Setup of Kempf–Ness type theorems

Let $(M, \omega, I, L, \|\cdot\|)$ be a **Hodge manifold**, i.e.

- (M, ω, I) Kähler manifold (not necessarily compact);
- $\omega = \frac{i}{2\pi} F$, F curvature of unitary holomorphic line bundle $L \to M$ with hermitian metric $\|\cdot\|$ (prequantization).

Example (standard)

- $M \subseteq \mathbb{CP}^n, \quad \omega = \omega_{\mathrm{FS}}|_M, \quad L = \mathcal{O}(1)|_M.$
- $M \subseteq \mathbb{C}^n, \quad \omega = \omega_{\mathrm{flat}}|_M, \quad L = M \times \mathbb{C}.$

Example (non-standard)

- Kodaira: compact + Hodge $\implies$ $M \subseteq \mathbb{CP}^n$ projective. But $\omega \neq \omega_{\mathrm{FS}}|_M$ in general.
- $M \subseteq \mathbb{C}^n$ with Kähler potential $f : M \to \mathbb{R}$, i.e. $\omega = 2i\partial\bar{\partial} f$. $\nexists$ isometry $M \hookrightarrow \mathbb{C}^N$ in general.

Setup of Kempf–Ness type theorems

Input

- $(M, \omega, I, L, \|\cdot\|)$ Hodge manifold, $L \to M$ complex algebraic;
- G compact Lie group;
- $G_{\mathbb{C}} \circlearrowright L$ such that G preserves $\|\cdot\|$.

Then, $G \circlearrowright M$ preserving (ω, I) and there is a **canonical moment map**

$$\mu : M \longrightarrow \mathfrak{g}^*, \quad \mu(p)(x) = \frac{d}{dt}\Big|_{t=0} \frac{1}{2\pi} \log \|e^{itx} \cdot \hat{p}\|,$$

for $x \in \mathfrak{g}$, $p \in M$, $\hat{p} \in L^* \setminus \{0\}$, $\hat{p} \mapsto p$.

Output

Two types of quotients:

1. **Symplectic quotient:** $\mu^{-1}(0)/G$ (stratified symplectic space)
2. **GIT quotient:** $M /\!\!/_L G_{\mathbb{C}}$ (complex algebraic variety)

Kempf–Ness type theorems

- We have $\mu^{-1}(0) \subseteq M^{L\text{-ss}}$, so there is a map

$$\mu^{-1}(0)/G \longrightarrow M /\!\!/_L G_{\mathbb{C}}. \tag{1}$$

- A **Kempf–Ness type theorem** is a condition which implies (1) is an isomorphism, i.e.
  - a homeomorphism respecting the natural stratifications;
  - the symplectic structures on the strata of the LHS and the complex structures on the strata of the RHS give Kähler structures.
- **Example.** M compact $\implies$ (1) is $\cong$.
  [Kirwan 1984] for the case $M \subseteq \mathbb{CP}^n$ with $\omega = \omega_{\mathrm{FS}}|_M$.
  [Sjamaar 1994] for the general case ($M \subseteq \mathbb{CP}^n$ but $\omega \neq \omega_{\mathrm{FS}}|_M$).
- If M is non-compact, we have to be more careful. We will discuss the case of affine varieties with $\omega = 2i\partial\bar{\partial}f$ in detail.

Complex analytic version of the Kempf–Ness theorem

First step: Complex analytic version.

$$\underbrace{M^{\mu\text{-ss}} := \{p \in M : \overline{G_{\mathbb{C}} \cdot p} \cap \mu^{-1}(0) \neq \emptyset\}}_{\text{analytically semistable points}} \underset{\substack{G_{\mathbb{C}}\text{-invariant}\\ \text{open}}}{\subseteq} M.$$

Theorem (Guillemin–Sternberg 1982, Kirwan 1984, Sjamaar 1994, Heinzner–Loose 1994)

There is a categorial quotient in the category of complex analytic spaces for $G_{\mathbb{C}} \circlearrowright M^{\mu\text{-ss}}$, denoted $M^{\mu\text{-ss}} /\!/ G_{\mathbb{C}}$. Moreover,

$$\begin{array}{ccc} \mu^{-1}(0) & \hookrightarrow & M^{\mu\text{-ss}} \\ \downarrow & & \downarrow \\ \mu^{-1}(0)/G & \xrightarrow{\cong} & M^{\mu\text{-ss}} /\!/ G_{\mathbb{C}}. \end{array}$$

Complex analytic version of the Kempf–Ness theorem

- **Recall:** GIT quotient

$$M /\!\!/_L G_{\mathbb{C}} = \underset{\substack{\text{categorical quot.} \\ \textbf{algebraic varieties}}}{M^{L\text{-ss}} /\!\!/ G_{\mathbb{C}}}.$$

- Luna 1976: Underlying complex analytic space

$$M /\!\!/_L G_{\mathbb{C}} = \underset{\substack{\text{categorical quot.} \\ \textbf{complex spaces}}}{M^{L\text{-ss}} /\!\!/ G_{\mathbb{C}}}.$$

- By previous theorem,

$$\mu^{-1}(0)/G \cong \underset{\substack{\text{categorical quot.} \\ \textbf{complex spaces}}}{M^{\mu\text{-ss}} /\!\!/ G_{\mathbb{C}}}$$

so, by uniqueness of categorical quotients, Kempf–Ness holds if

$$\boxed{M^{\mu\text{-ss}} = M^{L\text{-ss}}}$$

analytic semistability = algebraic semistability

The general Kempf–Ness theorem

Theorem (Kempf–Ness 1979, Mumford, Guillemin–Sternberg 1982, Ness 1984, Kirwan 1984, Sjamaar 1994, Heinzner–Loose 1994, ...)

- $(M, \omega, I, L, \|\cdot\|)$ *Hodge manifold*
- $G_{\mathbb{C}} \circlearrowright L$, *$G$ preserves* $\|\cdot\|$

Then, $G \circlearrowright (M, \omega, I)$ with canonical moment map $\mu : M \to \mathfrak{g}^*$. We have $\mu^{-1}(0) \subseteq M^{L\text{-ss}}$ so there is a map

$$\mu^{-1}(0)/G \longrightarrow M /\!\!/_L G_{\mathbb{C}}. \tag{2}$$

Suppose:

(i) **Algebraic Condition:** *(M, L) satisfies the* **geometric criterion***:*

$$M^{L\text{-ss}} = \{p \in M : \exists \hat{p} \in L^* \setminus \{0\}, \hat{p} \mapsto p, \overline{G_{\mathbb{C}} \cdot \hat{p}} \subseteq L^* \setminus \{0\}\}$$

e.g. M is projective, affine, or projective-over-affine.

(ii) **Analytic Condition:** $\|\cdot\|^2 : L^* \to \mathbb{R}$ *is proper on closed $G_{\mathbb{C}}$-orbits disjoint from the zero-section.*

Then, $M^{\mu\text{-ss}} = M^{L\text{-ss}}$ so (2) is an isomorphism.

Example. M compact $\implies$ (i) & (ii). So $\mu^{-1}(0)/G \cong M /\!\!/_L G_{\mathbb{C}}$.

The case of affine varieties

(I) Kempf–Ness 1979

- $M \subseteq \mathbb{C}^n$ complex affine
- $G_{\mathbb{C}} \circlearrowright M$ via $G_{\mathbb{C}} \to \mathrm{GL}(n, \mathbb{C})$
- $\omega = \omega_{\mathrm{flat}}|_M$
- $L = M \times \mathbb{C}, \quad G_{\mathbb{C}} \circlearrowright L, \quad g \cdot (p, z) = (g \cdot p, z).$

$\implies \mu = \mu_{\mathrm{std}}$

$$\mu_{\mathrm{std}} : M \longrightarrow \mathfrak{g}^*, \quad \mu_{\mathrm{std}}(p)(x) = -\frac{1}{2}\mathrm{Im}\langle xp, p\rangle, \quad (p \in M, x \in \mathfrak{g}).$$

Kempf–Ness holds, so

$$\mu_{\mathrm{std}}^{-1}(0)/G \cong \operatorname{Spec} \mathbb{C}[M]^{G_{\mathbb{C}}}$$

The case of affine varieties

(II) King 1994

- $M \subseteq \mathbb{C}^n$ complex affine
- $G_{\mathbb{C}} \circlearrowright M$ via $G_{\mathbb{C}} \to \mathrm{GL}(n, \mathbb{C})$
- $\omega = \omega_{\mathrm{flat}}|_M$
- $\boxed{L_\chi = M \times \mathbb{C}, \quad g \cdot (p, z) = (g \cdot p, \chi(g) z), \quad \chi : G_{\mathbb{C}} \to \mathbb{C}^*}$

$\implies \mu = \mu_{\mathrm{std}} - \xi$

$$\xi := \frac{i}{2\pi} d\chi \in \mathfrak{g}^*$$

Kempf–Ness holds, so

$$\mu^{-1}(\xi)/G \cong M /\!\!/_{L_\chi} G_{\mathbb{C}} = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \mathbb{C}[M]^{G_{\mathbb{C}}, \chi^n} \right)$$

The case of affine varieties

(III) Azad–Loeb 1993

- $M \subseteq \mathbb{C}^n$ complex affine
- $G_{\mathbb{C}} \circlearrowright M$ via $G_{\mathbb{C}} \to \mathrm{GL}(n, \mathbb{C})$
- $\boxed{\omega = 2i\partial\bar{\partial}f, \quad f : M \to \mathbb{R}, \quad G\text{-invariant}}$ $\qquad (f = \|\cdot\|^2$ recovers (I)).
- $L = M \times \mathbb{C}, \qquad g \cdot (p, z) = (g \cdot p, z)$

$\implies \mu = \mu_f$, where

$$\mu_f : M \longrightarrow \mathfrak{g}^*, \quad \mu_f(p)(x) = df(Ix_p^{\#}), \quad (p \in M, x \in \mathfrak{g}).$$

Kempf-Ness holds **if** f **is proper and bounded below**. In that case,

$$\mu_f^{-1}(0)/G \cong \operatorname{Spec} \mathbb{C}[M]^{G_{\mathbb{C}}}.$$

The case of affine varieties

(IV)

- $M \subseteq \mathbb{C}^n$ complex affine
- $G_{\mathbb{C}} \circlearrowright M$ via $G_{\mathbb{C}} \to \mathrm{GL}(n, \mathbb{C})$
- $\boxed{\omega = 2i\partial\bar{\partial}f, \quad f : M \to \mathbb{R}, \quad G\text{-invariant}}$
- $\boxed{L_\chi = M \times \mathbb{C}, \quad g \cdot (p, z) = (g \cdot p, \chi(g)z), \quad \chi : G_{\mathbb{C}} \to \mathbb{C}^*}$

$\implies \mu = \mu_f - \xi$

Kempf–Ness can fail even if f is proper and bounded below:

Example

$\mathbb{C}^* \circlearrowright \mathbb{C}^*$ with $f(z) = \sqrt{1 + (\log |z|^2)^2}$ and $\chi(z) = z^3$. Then,

$$\mu_f^{-1}(\xi)/G = \emptyset, \quad M /\!\!/_{L_\chi} G_{\mathbb{C}} = \{pt\}.$$

The case of affine varieties

Theorem

If

$$\boxed{\mathbb{C}[M] \subseteq o(e^f)}$$

i.e.

$$\forall \textit{ polynomial } u : M \to \mathbb{C}, \quad \lim_{p \to \infty} \frac{u(p)}{e^{f(p)}} = 0,$$

then the Kempf–Ness theorem holds, so

$$\mu_f^{-1}(\xi)/G \cong \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \mathbb{C}[M]^{G_{\mathbb{C}},\chi^n}\right),$$

where $\mu_f(p)(x) = df(Ix_p^{\#})$ *and* $\xi = \frac{i}{2\pi} d\chi \in \mathfrak{g}^*$.

For example, $\mathbb{C}[x] \subseteq o(x^{\log x}) = o(e^{(\log x)^2})$. The example with Nahm's equations will look like this, i.e. $f(x) \sim (\log |x|)^2$.

Example from Nahm's equations

Nahm's equations: 1D reduction of the self-dual Yang–Mills equations.

$$A = (A_0, A_1, A_2, A_3) : I \subseteq \mathbb{R} \longrightarrow \mathfrak{g} \otimes \mathbb{H}$$
$$\dot{A}_1 + [A_0, A_1] + [A_2, A_3] = 0$$
$$\dot{A}_2 + [A_0, A_2] + [A_3, A_1] = 0$$
$$\dot{A}_3 + [A_0, A_3] + [A_1, A_2] = 0.$$

Natural action by gauge transformations:

$$\mathcal{G} := \{g : I \to G\} \circlearrowleft \{\text{solutions to Nahm's eqs.}\}$$

- $I = [0, 1]$, $\mathcal{G}_0 = \{g \in \mathcal{G} : g(0) = g(1) = 1\}$.
- $\mathcal{M} := \{\text{solutions to Nahm's eqs}\}/\mathcal{G}_0$

Theorem (Kronheimer 1988)

- *$\mathcal{M}$ is a hyperkähler manifold; $(\mathcal{M}, g, I, J, K)$.*
- *$\mathcal{M} \cong T^* G_{\mathbb{C}}$, biholomorphism with respect to I.*

Example from Nahm's equations

Theorem (Dancer–Swann 1996)

- *$G \times G \circlearrowright T^*G_{\mathbb{C}}$ preserves hyperkähler structure.*
- *There is a hyperkähler moment map*

$$\mu : T^*G_{\mathbb{C}} \longrightarrow (\mathfrak{g}^* \times \mathfrak{g}^*)^3, \quad \mu(A) = \begin{pmatrix} A_1(0) & A_2(0) & A_3(0) \\ -A_1(1) & -A_2(1) & -A_3(1) \end{pmatrix}.$$

For all closed subgroup $H \subseteq G \times G$ and $\chi_1, \chi_2, \chi_3 : H \to S^1$,

$$T^*G_{\mathbb{C}} /\!/\!/_{\xi} H := \mu_{\mathfrak{h}}^{-1}(\xi)/H$$

is a stratified hyperkähler space, where $\xi = \frac{i}{2\pi}(d\chi_1, d\chi_2, d\chi_3) \in (\mathfrak{h}^*)^3$ and

$$\mu_{\mathfrak{h}} : \mathcal{M} \xrightarrow{\mu} (\mathfrak{g}^* \times \mathfrak{g}^*)^3 \xrightarrow{i_{\mathfrak{h}}^*} (\mathfrak{h}^*)^3.$$

Example from Nahm's equations

- $T^*G_\mathbb{C}$ is a complex affine variety
- $\nexists$ isometric $T^*G_\mathbb{C} \hookrightarrow \mathbb{C}^N$ in general
- $\omega_1 = 2i\partial\bar{\partial}f$ and $\mu_1 = \mu_f$, where

$$f : T^*G_\mathbb{C} \longrightarrow \mathbb{R}, \quad f(A) = \frac{1}{4}\int_0^1 \left(2\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2\right)$$

- $\mu_\mathbb{C} := \mu_2 + i\mu_3 : T^*G_\mathbb{C} \to \mathfrak{g}_\mathbb{C}^* \times \mathfrak{g}_\mathbb{C}^*$ is complex algebraic

Theorem

We have

$$\boxed{\mathbb{C}[T^*G_\mathbb{C}] \subseteq o(e^f).}$$

Hence, for all $H \subseteq G \times G$ *and* $\chi_1, \chi_2, \chi_3 : H \to S^1$,

$$T^*G_\mathbb{C} /\!\!/\!\!/_\xi H \cong \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} \mathbb{C}[\mu_\mathbb{C}^{-1}(\xi_2 + i\xi_3)]^{H_\mathbb{C}, \chi_1^n}\right).$$