Symplectic reduction and its application to Kähler and Hyperkähler geometries

Maxence Mayrand

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1 Introduction

The main goal of this essay is to prove the *Kähler quotient construction* (Theorem 6.5), which says that if a Kähler manifold M is acted on by a compact Lie group G preserving the Kähler structure and there is a moment map $\mu : M \to \mathfrak{g}^*$ for the action, then for all $\xi \in \mathfrak{g}^*$ such that the stabilizer G_ξ acts freely on $\mu^{-1}(\xi)$, the quotient space $\mu^{-1}(\xi)/G_\xi$ is also Kähler. We will also prove the Hyperkähler analogue of this construction.

The essay is organized as follows. In Section 2 and 3 we recall some basic facts from the theory of Lie groups and principal bundles. In Section 4 we describe the classical symplectic reduction of Marsden and Weinstein [7]. In Section 5 we introduce the *shifting trick* which allows to identify the symplectic reduction at $\xi \in \mathfrak{g}^*$ with the symplectic reduction at 0 of $M \times (G \cdot \xi)$ using a certain canonical symplectic structure on the coadjoint orbit $G \cdot \xi$. In Sections 6 and 7, following Hitchin, Karlhede, Lindström and Roček [3], we discuss how symplectic reduction applies to Kähler and Hyperkähler manifolds, respectively. Finally, we give in Section 8 an example of Kähler reduction. The emphasis throughout this essay is on providing detailed proofs of all results.

2 Lie Group Actions and Quotient Manifolds

Let M be a smooth manifold with a smooth left action of a compact Lie group G. Denote the action by

$$\psi : G \times M \longrightarrow M, \quad (g, p) \longmapsto g \cdot p.$$

We will also often use the notations

$$\psi(g, p) = \psi_g(p) = \psi^p(g) = g \cdot p,$$

and call

$$\psi^p : G \longrightarrow M, \quad g \longmapsto g \cdot p$$

the **orbit map**. We denote by G_p the stabilizer of G at p and by $G \cdot p$ the orbit through p. Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M and let

$$\widehat{\psi} : \mathfrak{g} \longrightarrow \mathfrak{X}(M), \quad X \longmapsto X^\# \tag{2.1}$$

be the Lie algebra action generated by ψ. That is, for each $X \in \mathfrak{g}$, $X^\#$ is the unique vector field on M whose flow is $(t, p) \mapsto \exp(tX) \cdot p$. Recall that $\widehat{\psi}$ is a Lie algebra anti-homomorphism, meaning that $[X, Y]^\# = -[X^\#, Y^\#]$. Also, we will often use that $X^\#$ can be expressed in terms of the orbit map by

$$X^\#_p = (d\psi^p)_e(X_e).$$

Moreover, we recall the following basic fact, whose proof can be found, for example, in [5] Theorem 21.10.

Theorem 2.1. *Let G be a Lie group acting smoothly, freely and properly on a smooth manifold M. Then, the quotient space M/G is a manifold and has a unique smooth structure such that the natural projection $\pi : M \to M/G$ is a smooth submersion. Moreover, $\pi : M \to M/G$ has the structure of a principal G-bundle.*

In our case, we do not assume that the action of G on M is free so this does not apply directly. Nevertheless, we can still use the theorem to construct a smooth manifold structure on the set G/G_p of left cosets. Then, we show that the orbits $G \cdot p$ are smooth submanifolds of M diffeomorphic to G/G_p.

Proposition 2.2. *The stabilizer G_p of a point $p \in M$ is a closed Lie subgroup of G. Moreover, the right action of G_p on G by right multiplication gives a compact smooth manifold G/G_p.*

Proof. We have $G_p = (\psi^p)^{-1}(p)$, so G_p is a closed subgroup of G and hence an embedded Lie subgroup of G (by the "closed subgroup theorem"). Now, G_p acts smoothly and freely on G by right multiplication. Moreover, G_p is closed in G so it is compact and hence the action is proper. Therefore, the orbit space G/G_p is a smooth manifold which is compact since $G/G_p = \pi(G)$ and G is compact. □

Proposition 2.3. *Let G be a compact Lie group acting smoothly on a smooth manifold M. Then, the orbit $G \cdot p$ through a point $p \in M$ is a properly embedded submanifold of M diffeomorphic to G/G_p. Moreover, the restriction of the orbit map to $G \cdot p$,*

$$\psi^p : G \longrightarrow G \cdot p, \quad g \longmapsto g \cdot p,$$

is a surjective smooth submersion and its derivative at e induces a surjective linear map

$$\mathfrak{g} \longrightarrow T_p(G \cdot p), \quad X \longmapsto X_p^{\#},$$

whose kernel is $\mathfrak{g}_p := \mathrm{Lie}(G_p)$. Thus, it is a linear isomorphism if $G_p = \{e\}$, and in general we have

$$T_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{g}_p.$$

Proof. Since G/G_p is the quotient of G by *right* multiplication of elements of G_p, the following left action of G on G/G_p is well-defined,

$$\varphi : G \times G/G_p \longrightarrow G/G_p, \quad g \cdot [h] = [gh].$$

Moreover, it is transitive since left multiplication by G on itself is transitive. To show that it is smooth, note that the following diagram commutes:

$$\begin{array}{ccccccc} G \times G & \xrightarrow{m} & G & \qquad & (g,h) & \longmapsto & gh \\ \downarrow{\scriptstyle Id\times\pi} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ G \times G/G_p & \xrightarrow{\varphi} & G/G_p & & (g,[h]) & \longmapsto & [gh] \end{array}$$

The upper right corner is smooth and $Id \times \pi$ is a smooth submersion, so φ is smooth.

Now, the orbit map $\psi^p : G \to M$ is constant on the fibres of $\pi : G \to G/G_p$ since if $g \in G$ and $a \in G_p$ then $\psi^p(ga) = g \cdot a \cdot p = g \cdot p = \psi^p(g)$. But π is a smooth submersion, so ψ^p descends to a smooth map

$$\Psi^p : G/G_p \longrightarrow M, \quad [h] \longmapsto h \cdot p.$$

It is injective since if $h_1 \cdot p = h_2 \cdot p$ then $h_2^{-1}h_1 \in G_p$ and hence $[h_1] = [h_2] \in G/G_p$. Moreover, the image of Ψ^p is the orbit $G \cdot p$. Now, Ψ^p is equivariant with respect to the transitive action of G on G/G_p and the original action of G on M, so it has constant rank. Since it is injective, it is a smooth immersion. Moreover, G/G_p is compact so Ψ^p is a smooth embedding and a proper map. Therefore, $\Psi^p(G/G_p) = G \cdot p$ is a properly embedded submanifold of M diffeomorphic to G/G_p.

Now, the orbit map can be rewritten as

$$\psi^p : G \xrightarrow{\pi} G/G_p \xrightarrow{\Psi^p} G \cdot p.$$

This is the composition of a smooth submersion with a diffeomorphism, so it is a smooth submersion. Hence,

$$(d\psi^p)_e : \mathfrak{g} \cong T_eG \longrightarrow T_p(G \cdot p), \quad X \longmapsto X_p^{\#}$$

is a surjective linear map. Now, we have $G_p = (\psi^p)^{-1}(p)$ and ψ^p is a smooth submersion, so $\ker(d\psi^p)_e = T_eG_p \cong \mathfrak{g}_p$. $\square$

Proposition 2.4. *We have*

$$T_p(G \cdot p) = \ker d\pi_p$$

and

$$T_{\pi(p)}(M/G) \cong T_pM/T_p(G \cdot p)$$

for all $p \in M$.

Proof. The map $\pi : M \to M/G$ is a smooth submersion, so

$$d\pi_p : T_pM \longrightarrow T_{\pi(p)}(M/G)$$

is a surjective linear map. Moreover, $G \cdot p$ is the submanifold defined by the level set $\pi^{-1}(\pi(p))$ so $T_p(G \cdot p) = \ker d\pi_p$. Hence, $d\pi_p$ factors as an isomorphism $T_pM/T_p(G \cdot p) \longrightarrow T_{\pi(p)}(M/G)$. $\square$

3 Principal Bundles and Connections

In this section we recall a few basic facts from the theory of connections on principal bundles. Our main reference for this material is [4].

Let G be a Lie group and

$$\pi : M \longrightarrow M/G$$

a principal G-bundle. Let $V \subseteq TM$ be the **vertical bundle**, i.e. $V_p = \ker d\pi_p$ for all $p \in M$. Then, V is a G-invariant subbundle of TM and a **Ehresmann connection** on M is a choice of another G-invariant subbundle H of TM such that $TM = V \oplus H$. We call H the **horizontal bundle**.

One of the important properties of the horizontal bundle is that $d\pi : TM \to T(M/G)$ restricts to isomorphisms

$$d\pi_p : H_p \xrightarrow{\sim} T_{\pi(p)}(M/G).$$

This allows us to identify smooth vector fields on M/G with G-invariant sections of H. This correspondence will be very important in the following sections, so we state it more precisely in the following proposition (which also introduces some notation and terminology). This is all standard material and can be found for example in [4] Chapter II.

Proposition 3.1 (Horizontal Lift)**.** *Let $\pi : M \to M/G$ be a principal bundle with a Ehresmann connection $TM = V \oplus H$. Every vector field $X \in \mathfrak{X}(M/G)$ has a unique **horizontal lift** X^*. That is, there exists a unique smooth G-invariant vector field $X^* \in \mathfrak{X}(M)$ such that*

(a) $X_p^* \in H_p$, and

(b) $d\pi_p(X_p^*) = X_{\pi(p)}$,

for all $p \in M$.

Conversely, given a smooth G-invariant section Y of H there is a unique vector field $X \in \mathfrak{X}(M/G)$ such that $X^* = Y$. We denote this vector field by $\pi_*(Y)$.

Thus we have a bijection

$$\begin{aligned} \mathfrak{X}(M/G) &\longleftrightarrow \Gamma(M,H)^G \\ X &\longmapsto X^* \\ \pi_*(Y) &\longleftarrow\!\shortmid Y. \end{aligned}$$

Moreover, this bijection is C^∞-linear, in the sense that

$$(fX)^* = (f \circ \pi)X^*, \quad (X+Y)^* = X^* + Y^*$$

and

$$\pi_*((f \circ \pi)Y) = f\pi_*(Y), \quad \pi_*(X+Y) = \pi_*(X) + \pi_*(Y)$$

for all $f \in C^\infty(M/G)$ and smooth vector fields X, Y on M/G. □

Proposition 3.2. *Let ω be G-invariant covariant k-tensor on M. Then, there is a unique covariant k-tensor η on M/G such that*

$$\eta(X_1, \ldots, X_k) \circ \pi = \omega(X_1^*, \ldots, X_k^*),$$

for all smooth vector fields $X_1, \ldots, X_k$ on M/G.

Proof. For simplicity of notation, we assume that ω is a 1-form. Let X be a smooth vector field on M/G. By the G-invariance of ω the map $\omega(X^*) : M \to \mathbb{R}$ is constant on the fibres of π. Indeed, if $\pi(p) = \pi(q)$ then $q = g \cdot p$ for some $g \in G$ and since X^* is also G-invariant we get

$$\omega_q(X_q^*) = \omega_q((d\psi_g)_p(X_p^*)) = (\psi_g^*\omega)_p(X_p^*) = \omega_p(X_p^*).$$

Since π is a surjective smooth submersion, there is thus a unique smooth map $\eta(X)$ such that the following diagram commutes

$$\begin{tikzcd}
M \arrow[d, "\pi"] \arrow[dr, "\omega(X^*)"] & \\
M/G \arrow[r, "\eta(X)"] & \mathbb{R}.
\end{tikzcd}$$

Moreover, the C^∞-linearity property of horizontal lifts shows that this defines a covariant tensor field η. □

Now, in the case where M has a G-invariant Riemannian metric g, there is a particular choice of connection, called the **orthogonal connection**, defined by taking H to be the orthogonal complement of V. The next result tells us that M/G inherits a metric whose Levi-Civita connection is easily computed in terms of the orthogonal projection $TM \to H$.

Theorem 3.3 (Metric on Quotient Manifolds)**.** *Let $\pi : M \to M/G$ be a principal G-bundle and g a G-invariant Riemannian metric on M. There exists a unique Riemannian metric $\bar{g}$ on M/G such that $\bar{g}(X,Y) \circ \pi = g(X^*, Y^*)$. Moreover, the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ on M/G is given by*

$$\bar{\nabla}_X Y = \pi_*(P_H(\nabla_{X^*} Y^*)), \tag{3.1}$$

where $P_H : V \oplus H \to H$ is the orthogonal projection and π_* is the push-forward of G-invariant horizontal sections given by Proposition 3.1.

The metric $\bar{g}$ is called the **quotient metric** induced by g.

Proof. By Proposition 3.2, the G-invariance of g implies that there exists a unique covariant 2-tensor $\bar{g}$ on M/G such that $\bar{g}(X,Y) \circ \pi = g(X^*, Y^*)$. It is clearly symmetric and positive definite and so a Riemannian metric.

Now, we show that (3.1) is a well-defined linear connection that is torsion free and compatible with $\bar{g}$. First, $P_H(\nabla_{X^*}Y^*)$ is G-invariant since X^*, Y^* are G-invariant, ∇ is compatible with the G-invariant metric g and H is G-invariant. Thus, its push-forward to M/G is well-defined by Proposition 3.1. Moreover, by the C^∞ linear property of horizontal lifts (Proposition 3.1), it is easy to see that $\bar{\nabla}$ is a linear connection. Thus, it remains to show that $\bar{\nabla}$ is torsion free and compatible with $\bar{g}$.

To show that $\bar{\nabla}$ is compatible with $\bar{g}$, let X, Y, Z be smooth vector fields on M/G. We have

$$\begin{aligned} Z_{\pi(p)}(\bar{g}(X,Y)) &= d(\bar{g}(X,Y))_{\pi(p)}(Z_{\pi(p)}) = d(\bar{g}(X,Y))_{\pi(p)}(d\pi_p(Z_p^*)) \\ &= d(g(X^*,Y^*))_p(Z_p^*) = Z_p^*(g(X^*,Y^*)), \end{aligned}$$

and hence

$$Z(\bar{g}(X,Y)) \circ \pi = Z^*(g(X^*,Y^*)) = g(\nabla_{Z^*}X^*,Y^*) + g(X^*,\nabla_{Z^*}Y^*).$$

Now, since X^* and Y^* are horizontal we have

$$\begin{aligned} Z(\bar{g}(X,Y)) \circ \pi &= g(P_H(\nabla_{Z^*}X^*),Y^*) + g(X^*,P_H(\nabla_{Z^*}Y^*)) \\ &= \bar{g}(\pi_*(P_H(\nabla_{Z^*}X^*)),Y) \circ \pi + \bar{g}(X,\pi_*(P_H(\nabla_{Z^*}Y^*))) \circ \pi \\ &= (\bar{g}(\bar{\nabla}_Z X,Y) + \bar{g}(X,\bar{\nabla}_Z Y)) \circ \pi. \end{aligned}$$

Thus, $Z(\bar{g}(X,Y)) = \bar{g}(\bar{\nabla}_Z X,Y) + \bar{g}(X,\bar{\nabla}_Z Y)$ so $\bar{\nabla}$ is compatible with $\bar{g}$.

Now to prove that $\bar{\nabla}$ is torsion free, recall that the horizontal component of $[X^*,Y^*]$ is $[X,Y]^*$ ([4], Proposition II.1.3). Thus,

$$\begin{aligned} \bar{\nabla}_X Y - \bar{\nabla}_Y X &= \pi_*(P_H(\nabla_{X^*}Y^* - \nabla_{Y^*}X^*)) = \pi_*(P_H([X^*,Y^*])) \\ &= \pi_*([X,Y]^*) = [X,Y]. \end{aligned}$$

□

The following standard fact will also be used later.

Theorem 3.4 (Metric on Submanifolds)**.** *Let (M,g) be a Riemannian manifold and $\tilde{M} \subseteq M$ an embedded submanifold with the induced Riemannian metric $\tilde{g} = i^*g$. Then, the Levi-Civita connection $\tilde{\nabla}$ of $(\tilde{M},\tilde{g})$ is given by*

$$\tilde{\nabla}_X Y = (\nabla_X Y)^\top,$$

where the vector fields on the right are extensions of $X, Y \in \mathfrak{X}(\tilde{M})$ to a neighborhood of $\tilde{M}$, and $\top$ is the orthogonal projection onto $T\tilde{M}$.

Proof. This is also known as Gauss Formula. See [6], Theorem 8.3. □

4 Hamiltonian Spaces and Symplectic Reduction

Let (M,ω) be a symplectic manifold and G a compact Lie group acting smoothly on M by symplectomorphisms.

We showed in Theorem 3.3 that when M is a Riemannian manifold and G acts isometrically, the quotient space M/G inherits a natural Riemannian structure. Symplectic reduction is the symplectic analogue of this phenomenon. However, here the situation is more subtle and we have to add extra assumptions. For one thing, M/G does not always have the right dimension to be symplectic. However, if we take a G-invariant submanifold S of codimension $\dim G$ in M, then $\dim S/G = \dim M - 2\dim G$ is even, so has the potential to be symplectic. One way to obtain such a submanifold, is by using a **moment map** for the action, i.e. a smooth map $\mu : G \to \mathfrak{g}^*$ with the following properties.

(1) μ is **equivariant**:

$$\mu(g \cdot p) = \mathrm{Ad}_g^*\mu(p), \qquad \text{for all } p \in M \text{ and } g \in G,$$

where Ad^* is the coadjoint representation of G on $\mathfrak{g}^*$.

(2) For each $X \in \mathfrak{g}$ define the function

$$\mu^X : M \longrightarrow \mathbb{R}, \quad \mu^X(p) = \langle \mu(p), X \rangle.$$

Then, μ^X is a **Hamiltonian function** for $X^\#$, i.e. $d\mu^X = i_{X^\#}\omega$.

It can be shown [2] that if G is semisimple, a moment map necessarily exists. When a Lie group G acts on a symplectic manifold (M, ω) by symplectomorphism and has a moment map μ, we call (M, ω, G, μ) a **Hamiltonian space**.

In particular, the equivariance of μ shows that for any $\xi \in \mathfrak{g}^*$, the stabilizer G_ξ preserves the level set $\mu^{-1}(\xi)$. Hence, it makes sense to consider the quotient $\mu^{-1}(\xi)/G_\xi$, and we wish to show that this space inherits a natural symplectic structure from the one on M.

First of all, by Proposition 2.3, G_ξ is a compact Lie subgroup of G and acts smoothly on M. To show that $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold, we want to use Theorem 2.1, so we want $\mu^{-1}(\xi)$ to be an embedded submanifold of M and G_ξ to act freely on $\mu^{-1}(\xi)$. But remarkably, the condition that G_ξ acts freely on $\mu^{-1}(\xi)$ automatically implies that ξ is a regular value of μ. In fact, we will show the following.

Lemma 4.1 (Conditions for Symplectic Reduction)**.** *The following are equivalent:*

(1) G_ξ *acts freely on* $\mu^{-1}(\xi)$.

(2) $G_p = \{e\}$ *for all* $p \in \mu^{-1}(\xi)$.

Moreover, if these hold, then ξ *is a regular value of* μ *and* $\mu^{-1}(\xi)/G_\xi$ *is a smooth manifold.*

Here (1) is just an algebraic condition which does not assume any topological or differential structure on $\mu^{-1}(\xi)$. As we will see at the end of this section, this condition is also sufficient to ensure the existence of a symplectic form on $\mu^{-1}(\xi)/G_\xi$.

The proof that (1) and (2) are equivalent will be a consequence of the following simple observation.

Proposition 4.2. *If* $p \in \mu^{-1}(\xi)$, *then*

$$(G_\xi)_p = G_\xi \cap G_p = G_p.$$

Proof. We have,

$$(G_\xi)_p = \{g \in G : \mathrm{Ad}_g^*\xi = \xi \text{ and } g \cdot p = p\} = G_\xi \cap G_p \subseteq G_p.$$

For the other inclusion, let $g \in G_p$. Then, by equivariance

$$\mathrm{Ad}_g^*\xi = \mathrm{Ad}_g^*\mu(p) = \mu(g \cdot p) = \mu(p) = \xi,$$

so $g \in G_p \cap G_\xi$. □

To prove the rest of Lemma 4.1, the key element will be the following result which expresses $\operatorname{im} d\mu_p$ in terms of the Lie algebra of G_p.

Proposition 4.3. *For all* $p \in M$ *we have*

$$\ker d\mu_p = (T_p(G \cdot p))^{\omega_p}$$
$$\operatorname{im} d\mu_p = \mathfrak{g}_p^0,$$

where $(T_p(G \cdot p))^{\omega_p}$ *is the symplectic complement of* $T_p(G \cdot p)$ *in* T_pM *and* $\mathfrak{g}_p^0 \subseteq \mathfrak{g}^*$ *is the annihilator of* $\mathfrak{g}_p := \mathrm{Lie}(G_p) \subseteq \mathfrak{g}$

Proof. First note that $G \cdot p$ is a properly embedded submanifold of M by Proposition 2.3. By viewing $X \in \mathfrak{g}$ as a linear map $\mathfrak{g}^* \to \mathbb{R}$ and using that the derivative of a linear map is the map itself, we get

$$\omega_p(X_p^\#, v) = d\mu_p^X(v) = d(X \circ \mu)_p(v) = X(d\mu_p(v)) = \langle d\mu_p(v), X\rangle.$$

Moreover, $T_p(G \cdot p) = \{X_p^\# : X \in \mathfrak{g}\}$ by Proposition 2.3, so we have

$$\begin{aligned}\ker d\mu_p &= \{v \in T_pM : \langle d\mu_p(v), X\rangle = 0, \forall X \in \mathfrak{g}\} \\ &= \{v \in T_pM : \omega_p(X_p^\#, v) = 0, \forall X \in \mathfrak{g}\} \\ &= \{v \in T_pM : \omega_p(w, v) = 0, \forall w \in T_p(G \cdot p)\} \\ &= (T_p(G \cdot p))^{\omega_p}.\end{aligned}$$

Now, let $v \in T_pM$ be is arbitrary. By Proposition 2.3, the kernel of the map $X \mapsto X_p^\#$ is $\mathfrak{g}_p$, so for all $X \in \mathfrak{g}_p$ we have

$$\langle d\mu_p(v), X\rangle = \omega_p(X_p^\#, v) = \omega_p(0, v) = 0.$$

Hence $\operatorname{im} d\mu_p \subseteq \mathfrak{g}_p^0$. Moreover, we have

$$\begin{aligned}\dim \operatorname{im} d\mu_p &= \dim T_pM - \dim \ker d\mu_p = \dim T_pM - \dim(T_p(G \cdot p))^{\omega_p} \\ &= \dim T_p(G \cdot p) = \dim \mathfrak{g}/\mathfrak{g}_p = \dim \mathfrak{g}_p^0,\end{aligned}$$

so $\operatorname{im} d\mu_p = \mathfrak{g}_p^0$. ☐

Proof of Lemma 4.1 (Conditions for Symplectic Reduction). By definition, G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $(G_\xi)_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. But $(G_\xi)_p = G_p$ for $p \in \mu^{-1}(\xi)$ so this is equivalent to (2).

Now, suppose that (1) and (2) hold. Then, $\mathfrak{g}_p = 0$ for all $p \in \mu^{-1}(\xi)$, so by Proposition 4.3, $\operatorname{im} d\mu_p = \mathfrak{g}_p^0 = \mathfrak{g}^*$ for all $p \in \mu^{-1}(\xi)$. Hence, ξ is a regular value of μ, so $\mu^{-1}(\xi)$ is a properly embedded submanifold of M and the action of G_ξ on M restricts to a smooth action on $\mu^{-1}(\xi)$. Since the action is free and G_ξ is compact, the quotient $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold by Theorem 2.1. ☐

The next result is the fundamental observation that will enable us to show that the quotient space $\mu^{-1}(\xi)/G_\xi$ admits a symplectic form.

Proposition 4.4. *Suppose that $\xi \in \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$.*

(a) *The level set $\mu^{-1}(\xi)$ is a properly embedded submanifold of M and*

$$T_p\mu^{-1}(\xi) = (T_p(G \cdot p))^{\omega_p}.$$

(b) *For each $p \in \mu^{-1}(\xi)$, the orbit $G_\xi \cdot p$ is a properly embedded submanifold of $\mu^{-1}(\xi)$ and*

$$T_p(G_\xi \cdot p) = (T_p\mu^{-1}(\xi))^{\omega_p} \cap T_p\mu^{-1}(\xi). \tag{4.1}$$

Proof. (a) By Lemma 4.1, ξ is a regular value of μ, so $\mu^{-1}(\xi)$ is a properly embedded submanifold of M. Moreover, by Proposition 4.3 we get

$$T_p\mu^{-1}(\xi) = \ker d\mu_p = (T_p(G \cdot p))^{\omega_p}.$$

(b) Let $p \in \mu^{-1}(\xi)$. We have $G_\xi \cdot p \subseteq \mu^{-1}(\xi)$ since if $g \in G_\xi$ then $\mu(g \cdot p) = \mathrm{Ad}_g^*\mu(p) = \mathrm{Ad}_g^*\xi = \xi$. Now, $G_\xi \cdot p$ is a properly embedded submanifold of M by Proposition 2.3. Since $G_\xi \cdot p \subseteq \mu^{-1}(\xi)$ and $\mu^{-1}(\xi)$ is an embedded submanifold of M, we get that $G_\xi \cdot p$ is a properly embedded submanifold of $\mu^{-1}(\xi)$.

In particular, $T_p(G_\xi \cdot p) \subseteq T_p\mu^{-1}(\xi)$. Now, by (a), we also have $T_p(G_\xi \cdot p) \subseteq T_p(G \cdot p) = (T_p\mu^{-1}(\xi))^{\omega_p}$, so

$$T_p(G_\xi \cdot p) \subseteq (T_p\mu^{-1}(\xi))^{\omega_p} \cap T_p\mu^{-1}(\xi).$$

To show the other inclusion, let

$$v \in (T_p\mu^{-1}(\xi))^{\omega_p} \cap T_p\mu^{-1}(\xi) = T_p(G \cdot p) \cap \ker d\mu_p.$$

Since $v \in T_p(G \cdot p)$ we have $v = X_p^\# = (d\psi^p)_e(X)$ for some $X \in \mathfrak{g}$ by Proposition 2.3. Let $\widehat{\mathrm{Ad}^*}(X)$ be the vector field on $\mathfrak{g}^*$ generated by X (in the notation of Equation (2.1) in Section 2). We have

$$(\mathrm{Ad}^*)^\xi(g) := \mathrm{Ad}_g^*\xi = \mathrm{Ad}_g^*\mu(p) = \mu(\psi_g(p)) = (\mu \circ \psi^p)(g), \quad \text{for all } g \in G,$$

so

$$\widehat{\mathrm{Ad}^*}(X)_\xi = d((\mathrm{Ad}^*)^\xi)_e(X) = d(\mu \circ \psi^p)_e(X) = d\mu_p(X_p^\#) = d\mu_p(v) = 0.$$

Thus, $X \in \mathfrak{g}_\xi = \mathrm{Lie}(G_\xi)$ by Proposition 2.3. But then $X_p^\# \in T_p(G_\xi \cdot p)$, so $v = X_p^\# \in T_p(G_\xi \cdot p)$. This proves Equation (4.1). □

Here is how I think about the previous result. We can always restrict the symplectic form ω to a 2-form $i^*\omega$ on $\mu^{-1}(\xi)$. However, this 2-form will have some degeneracies. In fact, if we have a non-degenerate 2-form Ω on a vector space V and restrict it to a subspace W, then the new form has kernel $W^\Omega \cap W$ and descends to a non-degenerate form on $W/(W^\Omega \cap W)$. But then Equation (4.1) tells us that the degeneracy of $i^*\omega$ occurs precisely in the direction of the G_ξ orbits! Thus, when we quotient the manifold by G_ξ we eliminate all degeneracies and get a genuine symplectic form. So most of the work is already done, it just remains to make that last argument a bit more precise.

Theorem 4.5 (Symplectic Reduction)**.** *Let (M, ω, G, μ) be a Hamiltonian space with G compact. Suppose that $\xi \in \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. Then, the orbit space $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold and there is a unique symplectic form ω_ξ on $\mu^{-1}(\xi)/G_\xi$ such that $\pi^*\omega_\xi = i^*\omega$ where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi$ and $i : \mu^{-1}(\xi) \hookrightarrow M$ are the natural maps. This symplectic space is denoted $M /\!/_\xi\, G_\xi$.*

Proof. By Lemma 4.1, $\mu^{-1}(\xi)$ is a smooth manifold on which the compact Lie group G_ξ acts smoothly and freely, giving a smooth manifold $\mu^{-1}(\xi)/G_\xi$ such that $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi$ is a principal G_ξ-bundle.

Now, the 2-form $i^*\omega$ on $\mu^{-1}(\xi)$ is G_ξ-invariant since if

$$\tilde{\psi} : G_\xi \times \mu^{-1}(\xi) \longrightarrow \mu^{-1}(\xi)$$

denotes the action of G_ξ on $\mu^{-1}(\xi)$, then $i \circ \tilde{\psi}_g = \psi_g \circ i$ for all $g \in G_\xi$, so

$$\tilde{\psi}_g^*(i^*\omega) = (i \circ \tilde{\psi}_g)^*\omega = (\psi_g \circ i)^*\omega = i^*\psi_g^*\omega = i^*\omega.$$

Thus, there is a unique 2-form ω_ξ on $\mu^{-1}(\xi)/G_\xi$ such that $\pi^*\omega_\xi = i^*\omega$. It is closed since $\pi^* d\omega_\xi = d(\pi^*\omega_\xi) = d(i^*\omega) = i^*d\omega = 0$ and $d\pi$ is surjective. It remains to show that ω_ξ is non-degenerate. Let $p \in \mu^{-1}(\xi)$ and $x = \pi(p) \in \mu^{-1}(\xi)/G_\xi$. Suppose that $v \in T_x(\mu^{-1}(\xi)/G_\xi)$ is such that $(\omega_\xi)_x(v, w) = 0$ for all $w \in T_x(\mu^{-1}(\xi)/G_\xi)$. Since the map

$$d\pi_p : T_p\mu^{-1}(\xi) \longrightarrow T_x(\mu^{-1}(\xi)/G_\xi)$$

is surjective, we have $v = d\pi_p(\hat{v})$ for some $\hat{v} \in T_p\mu^{-1}(\xi)$. Hence,

$$(i^*\omega)_p(\hat{v}, \hat{w}) = (\pi^*\omega_\xi)_p(\hat{v}, \hat{w}) = (\omega_\xi)_x(d\pi_p(\hat{v}), d\pi_p(\hat{w})) = 0$$

for all $\hat{w} \in T_p\mu^{-1}(\xi)$. Then, by Proposition 4.4,

$$\hat{v} \in (T_p\mu^{-1}(\xi))^{\omega_p} \cap T_p\mu^{-1}(\xi) = T_p(G_\xi \cdot p) = \ker d\pi_p.$$

Hence, $v = d\pi_p(\hat{v}) = 0$ so ω_ξ is non-degenerate. □

5 The Shifting Trick

In the context of the preceding section, we call an element $\xi \in \mathfrak{g}^*$ **central** if $G_\xi = G$, i.e. $\mathrm{Ad}_g^*\xi = \xi$ for all $g \in G$. In the next section, we will show that if ξ is central and the symplectic manifold (M, ω) possesses a G-invariant Kähler metric that is compatible with ω, then the quotient metric on $M/\!\!/_\xi\, G$ is also Kähler and compatible with the reduced form ω_ξ.

But there is a way to bypass the assumption that ξ is central by using what is called the "shifting trick". This allows us to identify $M/\!\!/_\xi\, G_\xi$ with the symplectic reduction at 0 of $M \times (G \cdot \xi)$ using a certain canonical symplectic structure on the coadjoint orbit $G \cdot \xi$. The purpose of this section is to explain this shifting procedure.

Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$. Denote by

$$\widehat{\mathrm{Ad}} : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathfrak{g}), \quad \text{and} \quad \widehat{\mathrm{Ad}^*} : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathfrak{g}^*)$$

the left $\mathfrak{g}$-actions generated by Ad and Ad^*, respectively (see Section 2, Equation (2.1)). It is a standard fact that $\widehat{\mathrm{Ad}}(X)_Y = [X, Y]$ and $\langle \widehat{\mathrm{Ad}^*}(X)_\xi, Y\rangle = \langle \xi, [Y, X]\rangle$ for all $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$.

Proposition 5.1. *For $\xi \in \mathcal{O}$, define a skew-symmetric bilinear form β_ξ on $\mathfrak{g}$ by*

$$\beta_\xi(X, Y) = \langle \xi, [X, Y]\rangle.$$

Then, $\mathfrak{g}^{\beta_\xi} = \mathfrak{g}_\xi$ where $\mathfrak{g}_\xi := \mathrm{Lie}(G_\xi)$.

Proof. Since $\langle \widehat{\mathrm{Ad}^*}(X)_\xi, Y\rangle = \langle \xi, [Y, X]\rangle = -\beta_\xi(X, Y)$, we have

$$\mathfrak{g}^{\beta_\xi} = \{X \in \mathfrak{g} : \langle \widehat{\mathrm{Ad}^*}(X)_\xi, Y\rangle = 0, \forall Y \in \mathfrak{g}\} = \{X \in \mathfrak{g} : \widehat{\mathrm{Ad}^*}(X)_\xi = 0\} = \mathfrak{g}_\xi,$$

by Proposition 2.2. □

Since G is a compact Lie group acting smoothly on $\mathfrak{g}^*$, Proposition 2.3 tells us that the orbit $\mathcal{O}$ is a properly embedded submanifold of $\mathfrak{g}^*$ with $T_\xi\mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\xi = \mathfrak{g}/\mathfrak{g}^{\beta_\xi}$. Thus, each bilinear form β_ξ descends to a non-degenerate 2-form on $T_\xi\mathcal{O}$ and it is easy to show that this defines a smooth G-invariant non-degenerate 2-form β on $\mathcal{O}$. Moreover, it is closed:

Proposition 5.2. *$d\beta = 0$.*

Proof. This is basically a consequence of the Jacobi identity. To simplify the notation, we let $X^\# = \widehat{\mathrm{Ad}^*}(X)$ be the vector field on $\mathfrak{g}^*$ generated by $X \in \mathfrak{g}$. Recall that

$$\begin{aligned} d\beta(X, Y, Z) = X(\beta(Y, Z)) - Y(\beta(X, Z)) + Z(\beta(X, Y)) \\ - \beta([X, Y], Z) + \beta([X, Z], Y) - \beta([Y, Z], X). \end{aligned} \tag{5.1}$$

By Proposition 2.3 the tangent space $T_\xi\mathcal{O}$ is generated by the vectors $X_\xi^\#$ for $X \in \mathfrak{g}$, so it suffices to compute $d\beta_\xi(X_\xi^\#, Y_\xi^\#, Z_\xi^\#)$ for $X, Y, Z \in \mathfrak{g}$ using the above formula. First, note that by the definition of β,

$$\beta(X^\#, Y^\#)_\xi = \beta_\xi(X, Y) = \langle \xi, [X, Y]\rangle,$$

so

$$\beta([X^\#, Y^\#], Z^\#)_\xi = \beta(-[X, Y]^\#, Z^\#)_\xi = -\langle \xi, [[X, Y], Z]\rangle.$$

Moreover, the function

$$\beta(Y^\#, Z^\#) : \mathfrak{g}^* \longrightarrow \mathbb{R}, \quad \xi \longmapsto \beta_\xi(Y, Z) = \langle \xi, [Y, Z]\rangle$$

is a linear map, so

$$X_\xi^\#(\beta(Y^\#, Z^\#)) = \langle X_\xi^\#, [Y, Z]\rangle = \langle \xi, [[Y, Z], X]\rangle.$$

Therefore, by Equation (5.1), we get

$$\begin{aligned} d\beta_\xi(X_\xi^\#, Y_\xi^\#, Z_\xi^\#) &= \langle \xi, [[Y,Z],X] - [[X,Z],Y] + [[X,Y],Z] \rangle \\ &\quad + \langle \xi, [[X,Y],Z] - [[X,Z],Y] + [[Y,Z],X] \rangle \\ &= -2\langle \xi, [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \rangle \\ &= 0 \end{aligned}$$

by the Jacobi identity. □

Putting all this together, we get the following result.

Theorem 5.3. *Let G be a compact Lie group. Each coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ is a properly embedded submanifold of $\mathfrak{g}^*$ and has a canonical G-invariant symplectic form $\beta_{\mathcal{O}}$.*

The 2-form $\beta_{\mathcal{O}}$ is sometimes called the **Kostant-Kirillov symplectic form**. Now, we need a moment map for the action of G on $(\mathcal{O}, \beta_{\mathcal{O}})$. It turns out that the moment map is the simplest one possible:

Proposition 5.4. *The coadjoint action of G on $(\mathcal{O}, \beta_{\mathcal{O}})$ is Hamiltonian with moment map the inclusion $\mu_{\mathcal{O}} : \mathcal{O} \hookrightarrow \mathfrak{g}^*$.*

Proof. We first show that $d\mu_{\mathcal{O}}^X = i_{X^\#}\beta$ for all $X \in \mathfrak{g}$. Note that

$$\mu_{\mathcal{O}}^X : \mathcal{O} \subseteq \mathfrak{g}^* \longrightarrow \mathbb{R}, \quad \mu_{\mathcal{O}}^X = X \circ \mu_{\mathcal{O}}$$

for $X \in \mathfrak{g}$, where we view X as a linear map $\mathfrak{g}^* \to \mathbb{R}$. Thus, for all $\eta \in \mathcal{O}$, we have

$$(d\mu_{\mathcal{O}}^X)_\eta : T_\eta\mathcal{O} \subseteq \mathfrak{g}^* \longrightarrow \mathbb{R},$$

is given by

$$(d\mu_{\mathcal{O}}^X)_\eta(w) = d(X \circ \mu_{\mathcal{O}})_\eta(w) = X \circ (d\mu_{\mathcal{O}})_\eta(w) = \langle w, X \rangle.$$

But $\# : \mathfrak{g} \to T_\eta\mathcal{O}$ is surjective, so it suffices to compute $(d\mu_{\mathcal{O}}^X)_\eta$ at $Y_\eta^\#$ for some $Y \in \mathfrak{g}$. We have

$$(d\mu_{\mathcal{O}}^X)_\eta(Y_\eta^\#) = \langle Y_\eta^\#, X \rangle = \langle \eta, [X,Y] \rangle = \beta_\eta(X,Y) = \beta(X_\eta^\#, Y_\eta^\#).$$

Thus, $d\mu_{\mathcal{O}}^X = i_{X^\#}\beta$. Equivariance is trivial since μ is the inclusion. □

Recall that a product $M_1 \times M_2$ of two symplectic manifolds (M_i, ω_i) is naturally a symplectic manifold with symplectic form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$, where π_i are the natural projections. Moreover, it is easy to show that if G is a Lie group acting on each M_i in a Hamiltonian way with moment maps $\mu_i : M_i \to \mathfrak{g}^*$, then the diagonal action of G on $M_1 \times M_2$ is Hamiltonian with moment map $\mu_1 \circ \pi_1 + \mu_2 \circ \pi_2$.

Now, we return to our original problem: We have a Hamiltonian space (M, ω, G, μ) with G compact. Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$. Then, by equipping $\mathcal{O}$ with the negative of its the Kostant-Kirillov form, we get a symplectic manifold $M \times \mathcal{O}$ on which G acts in a Hamiltonian way with moment map

$$\mu_{\mathcal{O}} : M \times \mathcal{O} \longrightarrow \mathfrak{g}^*, \quad (q, \eta) \longmapsto \mu(q) - \eta.$$

Thus it makes sense to consider the symplectic reduction of $M \times \mathcal{O}$ at 0. In fact, we will show that $(M \times \mathcal{O})/\!\!/_0\, G$ can be identified with $M/\!\!/_\xi\, G_\xi$ for any $\xi \in \mathcal{O}$. First, the next result tells us that the conditions of Theorem 4.5 (Symplectic Reduction) are satisfied for $M \times \mathcal{O}$ at 0 if and only if they are satisfied for M at some $\xi \in \mathcal{O}$.

Lemma 5.5. *Let $\xi \in \mathcal{O}$. Then, $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$ if and only if $G_{(p,\eta)} = \{e\}$ for all $(p, \eta) \in \mu_{\mathcal{O}}^{-1}(0)$.*

Proof. Suppose first that $G_{(p,\eta)} = \{e\}$ for all $(p,\eta) \in \mu_{\mathcal{O}}^{-1}(0)$. Let $p \in \mu^{-1}(\xi)$ and $g \in G_p$. Then,

$$\mathrm{Ad}_g^*\xi = \mathrm{Ad}_g^*\mu(p) = \mu(g \cdot p) = \mu(p) = \xi,$$

so $g \cdot (p,\xi) = (g \cdot p, \mathrm{Ad}_g^*\xi) = (p,\xi)$, and hence, $g \in G_{(p,\eta)} = \{e\}$.

Conversely, suppose that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. Let $(p,\eta) \in \mu_{\mathcal{O}}^{-1}(0)$ and $g \in G_{(p,\eta)}$. Then, $\eta = \mu(p) = \mathrm{Ad}_h^*\xi$ for some $h \in G$, and

$$g \cdot (p,\eta) = (g \cdot p, \mathrm{Ad}_g^*\eta) = (p,\eta).$$

Hence,

$$h^{-1}gh \cdot (h^{-1} \cdot p) = h^{-1} \cdot p,$$

so

$$h^{-1}gh \in G_{h^{-1}\cdot p}.$$

But $h^{-1} \cdot p \in \mu^{-1}(\xi)$ since

$$\mu(h^{-1} \cdot p) = \mathrm{Ad}_{h^{-1}}^*\mu(p) = \mathrm{Ad}_{h^{-1}}^*\eta = \xi.$$

Thus we get that $h^{-1}gh = e$ and hence $g = e$. □

We are now ready to prove that under these conditions, $(M \times \mathcal{O})/\!\!/_0\, G$ and $M/\!\!/_\xi\, G_\xi$ are symplectomorphic.

Theorem 5.6 (The Shifting Trick)**.** *Let (M,ω,G,μ) be a Hamiltonian space with G compact. Let $\xi \in \mathfrak{g}^*$ and let $\mathcal{O}$ be the coadjoint orbit through ξ. Then, $\mathcal{O}$ has a canonical symplectic form for which the diagonal action of G on $M \times \mathcal{O}$ is Hamiltonian with moment map $\mu_{\mathcal{O}}(q,\eta) = \mu(q) - \eta$. Moreover, $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$ if and only if $G_{(q,\eta)} = \{e\}$ for all $(q,\eta) \in \mu_{\mathcal{O}}^{-1}(0)$, and in that case, $M/\!\!/_\xi\, G_\xi$ and $(M \times \mathcal{O})/\!\!/_0\, G$ are well-defined and symplectomorphic.*

By Lemma 5.5 we have $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$ if and only if $G_{(q,\eta)} = \{e\}$ for all $(q,\eta) \in \mu_{\mathcal{O}}^{-1}(0)$ which are the conditions of Theorem 4.5 (Symplectic Reduction) ensuring that $M/\!\!/_\xi\, G_\xi$ and $(M \times \mathcal{O})/\!\!/_0\, G$ are both well-defined. Thus, it only remains to show that they are symplectomorphic. We have broken the proof into three steps.

Step 1. Let

$$F : \mu^{-1}(\xi) \longrightarrow \mu_{\mathcal{O}}^{-1}(0), \quad F(p) = (p,\xi).$$

Then, there is a smooth bijective map Φ such that the following diagram commutes:

$$\begin{tikzcd} \mu^{-1}(\xi) \arrow[r, "F"] \arrow[d, "\pi_1"] & \mu_{\mathcal{O}}^{-1}(0) \arrow[d, "\pi_2"] \\ \mu^{-1}(\xi)/G_\xi \arrow[r, "\Phi"] & \mu_{\mathcal{O}}^{-1}(0)/G \end{tikzcd}$$

Proof. For all $p \in \mu^{-1}(\xi)$ and $g \in G_\xi$, we have

$$F(g \cdot p) = (g \cdot p, \mu(g \cdot p)) = (g \cdot p, \mathrm{Ad}_g^*\mu(p)) = g \cdot (p, \mu(p)) = g \cdot F(p),$$

so F intertwines the actions. In particular, $\pi_2 \circ F$ is a smooth map which is constant on the fibres of π_1, and since π_1 is a smooth submersion, this implies that there exists a unique smooth map Φ such that the above diagram commutes.

Next, we show that Φ is bijective. Let $[(q,\eta)] \in \mu_{\mathcal{O}}^{-1}(0)/G$. Then, $\eta \in \mathcal{O}$ so there exists $g \in G$ such that $\mathrm{Ad}_g^*\eta = \xi$, and hence $\mu(g \cdot q) = \mathrm{Ad}_g^*\mu(q) = \mathrm{Ad}_g^*\eta = \xi$ so $g \cdot q \in \mu^{-1}(\xi)$. Thus, $\Phi([g \cdot q]) = \pi_2(F(g \cdot q)) = [(g \cdot q, \xi)] = [(q, \mathrm{Ad}_{g^{-1}}^*\xi)] = [(q,\eta)]$, so Φ is surjective. Now, suppose $\Phi([p]) = \Phi([q])$. Then, $\pi_2(p,\xi) = \pi_2(q,\xi)$ so there exists $g \in G$ such that $(p,\xi) = (g \cdot q, \mathrm{Ad}_g^*\xi)$. Thus, $p = g \cdot q$ for $g \in G_\xi$ so $[p] = [q] \in \mu^{-1}(\xi)/G$ and hence Φ is injective. □

Step 2. The map Φ is a diffeomorphism.

Proof. Since Φ is a smooth bijection, to show that it is a diffeomorphism it suffices to show that it is a smooth submersion. Since π_1 is a surjective smooth submersion, it suffices to show that $\pi_2 \circ F$ is a smooth submersion (indeed since $(d\pi_1)_p$ is surjective we get $\mathrm{rank}_{\pi(p)}(\Phi) = \mathrm{rank}_p(\Phi \circ \pi_1) = \mathrm{rank}_p(\pi_2 \circ F)$). We use the fact that a smooth map is a smooth submersion if and only if every point of the domain is in the image of a smooth local section. In other words, we let $p \in \mu^{-1}(\xi)$ and seek an open subset $W \subseteq \mu_{\mathcal{O}}^{-1}(0)$ with a smooth map $\rho : W \to \mu^{-1}(\xi)$ such that $(\pi_2 \circ F) \circ \rho = Id_W$ and $p \in \rho(W)$. By Proposition 2.3, the orbit map

$$G \longrightarrow \mathcal{O}, \quad g \longmapsto \mathrm{Ad}_g^* \xi$$

is a smooth submersion, so the characterization of smooth submersions in terms of local sections applies to this particular map and hence there exists an open set $V \subseteq \mathcal{O}$ with a smooth map $\varphi : V \to G$ such that $e \in \varphi(V)$ and

$$\mathrm{Ad}_{\varphi(\eta)}^* \xi = \eta, \quad \text{for all } \eta \in V.$$

Since $e \in \varphi(V)$, there is $\eta_0 \in V$ such that $e = \varphi(\eta_0)$. But then $\xi = \mathrm{Ad}_e^* \xi = \mathrm{Ad}_{\varphi(\eta_0)}^* \xi = \eta_0$. Thus, $\xi \in V$ and $\varphi(\xi) = e$. Now, let

$$U = \{(q, \eta) \in \mu_{\mathcal{O}}^{-1}(0) : \eta \in V\} = \mu_{\mathcal{O}}^{-1}(0) \cap (M \times V).$$

This an open subset of $\mu_{\mathcal{O}}^{-1}(0)$ since $M \times V$ is open in $M \times \mathcal{O}$ and $\mu_{\mathcal{O}}^{-1}(0)$ has the subspace topology inherited from $M \times \mathcal{O}$ since it is embedded. Now, for $(q, \eta) \in U$ we have $\mathrm{Ad}_{\varphi(\eta)}^* \xi = \eta = \mu(q)$, so $\mathrm{Ad}_{\varphi(\eta)^{-1}}^* \mu(q) = \xi$ and hence

$$\mu(\varphi(\eta)^{-1} \cdot q) = \mathrm{Ad}_{\varphi(\eta)^{-1}}^* \mu(q) = \xi.$$

Thus, we have a smooth map

$$\sigma : U \subseteq \mu_{\mathcal{O}}^{-1}(0) \longrightarrow \mu^{-1}(\xi), \quad (q, \eta) \longmapsto \varphi(\eta)^{-1} \cdot q.$$

We have $\mu(p) = \xi \in V$ so $(p, \xi) \in U$ and $\sigma(p, \xi) = \varphi(\xi)^{-1} \cdot p = e^{-1} \cdot p = p$. Thus, $p \in \sigma(U)$. Now, since π_2 is a smooth submersion, there exists an open set $W \subseteq \mu_{\mathcal{O}}^{-1}(0)/G$ and a smooth local section $\tau : W \to \mu_{\mathcal{O}}^{-1}(0)$ such that $(p, \xi) \in \tau(W)$. Then, $\tilde{W} := \tau^{-1}(U)$ is an open subset of W and hence we have a smooth map

$$\rho : \tilde{W} \xrightarrow{\tau} U \xrightarrow{\sigma} \mu^{-1}(\xi).$$

To show that it is a local section of $\pi_2 \circ F$ let $w \in \tilde{W}$. Then, $w \in W$ so $\tau(w) = (q, \eta)$ for some $(q, \eta) \in U \subseteq \mu_{\mathcal{O}}^{-1}(0)$ such that $\pi_2(q, \eta) = w$. Then,

$$\begin{aligned} ((\pi_2 \circ F) \circ \rho)(w) &= \pi_2(F(\sigma(q, \eta))) = \pi_2(F(\varphi(\eta)^{-1} \cdot q)) = \pi_2(\varphi(\eta)^{-1} \cdot q, \xi) \\ &= \pi_2(q, \mathrm{Ad}_{\varphi(\eta)}^* \xi) = \pi_2(q, \eta) = w. \end{aligned}$$

Hence, ρ is a smooth local section of $\pi_2 \circ F$. Moreover, $(p, \xi) \in \tau(W)$ so $(p, \xi) = \tau(w)$ for some $w \in W$. Also, $\tau(w) = (p, \xi) \in U$ so $w \in \tau^{-1}(U) = \tilde{W}$. Thus, $p = \sigma(p, \xi) = \sigma(\tau(w)) = \rho(w) \in \rho(\tilde{W})$. This concludes the proof that $\pi_2 \circ F$ is a smooth submersion, and hence that Φ is a diffeomorphism. ☐

Step 3. The map Φ is a symplectomorphism.

Proof. Let

$$\begin{tikzcd} & M \times \mathcal{O} \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M & & \mathcal{O} \end{tikzcd}$$

be the canonical projections. Let $\beta_{\mathcal{O}}$ be the Kostant-Kirillov form on $\mathcal{O}$. Then, by definition, the symplectic form on $M \times \mathcal{O}$ is

$$\alpha := \rho_1^*\omega - \rho_2^*\beta_{\mathcal{O}},$$

and the symplectic form on $(M \times \mathcal{O}) /\!\!/_0 G$ is the unique 2-form $\omega_{\mathcal{O}}$ such that $\pi_2^*\omega_{\mathcal{O}} = i_2^*\alpha$, where $i_2 : \mu_{\mathcal{O}}^{-1}(0) \hookrightarrow M \times \mathcal{O}$ is the inclusion map. Now, ω_ξ is the unique 2-form on $M /\!\!/_\xi G_\xi$ such that $\pi_1^*\omega_\xi = i_1^*\omega$, where $i_1 : \mu^{-1}(\xi) \hookrightarrow M$, so it suffices to that $\pi_1^*\Phi^*\omega_{\mathcal{O}} = i_1^*\omega$. We have

$$\begin{aligned}\pi_1^*\Phi^*\omega_{\mathcal{O}} &= (\Phi \circ \pi_1)^*\omega_{\mathcal{O}} = (\pi_2 \circ F)^*\omega_{\mathcal{O}} = F^*\pi_2^*\omega_{\mathcal{O}} = F^*i_2^*\alpha = F^*i_2^*(\rho_1^*\omega - \rho_2^*\beta_{\mathcal{O}}) \\ &= (\rho_1 \circ i_2 \circ F)^*\omega - (\rho_2 \circ i_2 \circ F)\beta_{\mathcal{O}}.\end{aligned}$$

Now, for all $p \in \mu^{-1}(\xi)$ we have

$$\rho_1 \circ i_2 \circ F(p) = \rho_1(p, \xi) = p = i_1(p),$$

so $(\rho_1 \circ i_2 \circ F)^*\omega = i_1^*\omega$, and

$$\rho_2 \circ i_2 \circ F(p) = \rho_2(p, \xi) = \xi,$$

so $(\rho_2 \circ i_2 \circ F)^*\beta_{\mathcal{O}} = 0$. Thus, $\pi_1^*\Phi^*\omega_{\mathcal{O}} = i_1^*\omega$, and hence $\Phi^*\omega_{\mathcal{O}} = \omega_\xi$. □

6 Kähler Reduction

Let us first recall the definition of a Kähler manifold.

Definition 6.1. Let X be a smooth manifold. An **almost Hermitian structure** on X is a triple (I, g, ω), where I is an almost complex structure, g is a Riemannian metric and ω is a 2-form such that

$$\omega(X, Y) = g(IX, Y), \quad \forall X, Y.$$

In that case, the triple is said to be **compatible**. An almost hermitian manifold (X, I, ω, g) is called **Kähler** if I is integrable and ω is closed.

The following standard result will be useful.

Theorem 6.2. *Let (X, I, g, ω) be an almost hermitian manifold. Then, the following are equivalent:*

(1) *(X, I, g, ω) is Kähler,*

(2) *$\nabla\omega = 0$,*

(3) *$\nabla I = 0$.*

We showed how to endow a symplectic form $\beta_{\mathcal{O}}$ on each coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. But there is also a way to give a Kähler metric on $\mathcal{O}$ whose Kähler form is $\beta_{\mathcal{O}}$.

Theorem 6.3. *Let G be a compact Lie group and $\mathcal{O} \subseteq \mathfrak{g}^*$ a coadjoint orbit. There is a unique G-invariant Kähler metric on $\mathcal{O}$ that is compatible with its Kostant-Kirillov form.*

Proof. See [1], Chapter 8. □

This result will be important when we use the shifting trick to show that $M /\!\!/_\xi G_\xi$ is Kähler by identifying it with $(M \times \mathcal{O}) /\!\!/_0 G$.

Now, a basis fact from Riemannian geometry:

Lemma 6.4. *Let (M, g) be a Riemannian manifold, and*

$$f : M \longrightarrow \mathbb{R}^k, \quad p \longmapsto (f_1(p), \ldots, f_k(p))$$

a smooth function with regular value $c \in \mathbb{R}^k$. Let $\tilde{M} = f^{-1}(c)$ be the induced submanifold of M. Then, $\{\mathrm{grad}(f_1), \ldots, \mathrm{grad}(f_k)\}$ is a smooth global frame for the normal bundle $N\tilde{M}$ over $\tilde{M}$.

Proof. Let $X_i = \operatorname{grad}(f_i)$. First, it is clear that $(X_i)_p \in N_p$ for all $p \in f^{-1}(c)$ since $g((X_i)_p, Y_p) = (df_i)_p Y_p = 0$ for all $Y_p \in T_p f^{-1}(c) = \ker df_p$. Moreover, since $f^{-1}(c)$ has dimension $n-k$, N is a bundle of rank $n-(n-k)=k$, so it suffices to show that $X_1, \dots, X_k$ are linearly independent at each point $p \in f^{-1}(c)$. By definition, $X_i = \operatorname{grad}(f_i)$ corresponds to df_i under the tangent-cotangent isomorphism provided by the metric g, so it suffices to that $(df_1)_p, \dots, (df_k)_p \in T_p^*M$ are linearly independent at each point $p \in f^{-1}(c)$. But this is equivalent to the condition that df_p is surjective. $\square$

We are now ready to state and prove the main theorem of this essay.

Theorem 6.5 (Kähler Reduction)**.** *Let (M, ω, g, I) be a Kähler manifold and G a compact Lie group acting on M isometrically and in a Hamiltonian way. Let μ be the moment map for this action and suppose that $\xi \in \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$.*

Then, the symplectic reduction $(M/\!\!/_\xi\, G_\xi, \omega_\xi)$ (whose existence is guaranteed by Theorem 4.5) admits a Kähler metric g_ξ that is compatible with ω_ξ. Moreover, if ξ is central, then g_ξ is the quotient metric induced by i^*g, where $i : \mu^{-1}(\xi) \hookrightarrow M$ is inclusion.

We have broken the proof into eight steps.

Step 1. By the shifting trick, it suffices to consider the case where ξ is central.

Proof. Suppose that the theorem holds for all central elements of $\mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be the coadjoint orbit through ξ. By Theorem 5.6 (The Shifting Trick), $M/\!\!/_\xi\, G_\xi$ is symplectomorphic to $(M \times \mathcal{O})/\!\!/_0\, G$. But $0 \in \mathfrak{g}^*$ is central and $M \times \mathcal{O}$ is Kähler (Theorem 6.3), so $(M \times \mathcal{O})/\!\!/_0\, G$ admits Kähler metric compatible with its symplectic form. Thus, we can use the symplectomorphism to get a Kähler metric on $M/\!\!/_\xi\, G_\xi$ that is compatible with ω_ξ. $\square$

Note, however, that in this case there is nothing that guarantees that the metric g_ξ so produced is the quotient metric induced by g. But this will be true if ξ is central.

Thus, from now on, we assume that ξ is central. In particular, $G_\xi = G$ so the reduced space is $M/\!\!/_\xi\, G = \mu^{-1}(\xi)/G$. Let

$$\pi : \mu^{-1}(\xi) \longrightarrow \mu^{-1}(\xi)/G$$

be the canonical projection. Let V be the vertical bundle of this principal G-bundle and let H be its orthogonal complement with respect to the metric i^*g on $\mu^{-1}(\xi)$. Let N be the normal bundle of the submanifold $\mu^{-1}(\xi) \subseteq M$. Then, for each $p \in \mu^{-1}(\xi)$ we have an orthogonal decomposition

$$T_pM = N_p \oplus V_p \oplus H_p.$$

Step 2. The horizontal bundle H is invariant under I.

Proof. First note that for $X \in \mathfrak{g}$, we have

$$g(\operatorname{grad} \mu^X, Y) = d\mu^X(Y) = \omega(X^\#, Y) = g(IX^\#, Y),$$

and hence

$$\operatorname{grad} \mu^X = IX^\#.$$

Now, let $X_1, \dots, X_k$ be a basis for $\mathfrak{g}$ and $\xi^1, \dots, \xi^k \in \mathfrak{g}^*$ its dual basis. Then,

$$\mu(p) = \mu^{X_1}(p)\,\xi^1 + \cdots + \mu^{X_k}(p)\,\xi^k, \quad \forall p \in M,$$

and hence, by Lemma 6.4, a global frame for the normal bundle N is

$$\{\operatorname{grad} \mu^{X_1}, \dots, \operatorname{grad} \mu^{X_k}\} = \{IX_1^\#, \dots, IX_k^\#\}.$$

Moreover, since $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$, Proposition 2.3 gives that $\{X_1^\#, \dots, X_k^\#\}$ is a basis for V_p. Therefore, at each point $p \in \mu^{-1}(\xi)$, the set

$$\{X_1^\#, \dots, X_k^\#, IX_1^\#, \dots, IX_k^\#\}$$

is a basis for $N_p \oplus V_p$. Hence, $N_p \oplus V_p$ is invariant under I. Now, let $v \in H_p$. Then, for all $w \in N_p \oplus V_p$ we have

$$g(I(v), w) = -g(v, I(w)) = 0$$

since $v \in H_p = (N_p \oplus V_p)^{\perp}$ and $I(w) \in N_p \oplus V_p$. Thus, $I(v) \in (N_p \oplus V_p)^{\perp} = H_p$ and hence H_p is preserved by I. $\square$

Note that for a smooth vector field X on $M /\!\!/_{\xi} G = \mu^{-1}(\xi)/G$, the horizontal lift X^* is not a vector field on M (only on $\mu^{-1}(\xi)$) so $I(X^*)$ is undefined in the usual sense. However, since I preserves H it does define a smooth section of H:

Step 3. For each smooth vector field X on $M /\!\!/_{\xi} G$, the map

$$\mu^{-1}(\xi) \longrightarrow H, \quad p \longmapsto I_p(X_p^*)$$

is a smooth G-invariant section of H which we denote by IX^*.

Proof. Since I preserves H, it defines a smooth bundle homomorphism $\bar{I} : H \to H$ and our map is the composition $\bar{I} \circ X^*$, so it is smooth. It is G-invariant since X^* is G-invariant and G preserves I. $\square$

Since IX^* is a G-invariant horizontal section, it is the lift of a unique smooth vector field $\pi_*(IX^*)$ on $M /\!\!/_{\xi} G$. Hence, we have a way of applying I to vector fields on $M /\!\!/_{\xi} G$. The next step is to show that this defines an almost complex structure on $M /\!\!/_{\xi} G$.

Step 4. There is an almost complex structure I_ξ on $M /\!\!/_{\xi} G$ defined by

$$I_\xi X = \pi_*(IX^*),$$

for all smooth vector fields X on $M /\!\!/_{\xi} G$.

Proof. To show that I_ξ is a type-$(1,1)$ tensor field, we have to show that it is C^∞ linear. This follows from the C^∞ linear property of horizontal lifts (Proposition 3.1), since

$$\begin{aligned} I_\xi(fX) &= \pi_*(I(fX)^*) = \pi_*(I((f \circ \pi)X^*)) = \pi_*((f \circ \pi)IX^*) = f\pi_*(IX^*) \\ &= fI_\xi(X). \end{aligned}$$

Now we need to show that $I_\xi^2 = -Id$. But, by definition we have $(I_\xi X)^* = IX^*$ so

$$(I_\xi^2 X)^* = (I_\xi(I_\xi(X)))^* = I((I_\xi(X))^*) = I(I(X^*)) = -X^* = (-X)^*,$$

and hence $I_\xi^2 X = -X$. $\square$

Step 5. Let g_ξ be the quotient metric on $M /\!\!/_{\xi} G$ induced by i^*g (Theorem 3.3). Then, the Levi-Civita connection of g_ξ is given by

$$\bar{\nabla}_X Y = \pi_*(\widehat{P}_H(\nabla_{X^*} Y^*))$$

where X^* and Y^* are extended arbitrarily to a neighborhood of $\mu^{-1}(\xi)$, ∇ is the Levi-Civita connection of g,

$$\widehat{P}_H : TM \longrightarrow H$$

is the orthogonal projection, and π_* is the push-forward of G-invariant horizontal sections.

Proof. Let $\tilde{\nabla}$ be the Levi-Civita connection of i^*g on $\mu^{-1}(0)$. Then, by Theorem 3.4, $\tilde{\nabla}$ is given explicitely by

$$\tilde{\nabla}_X Y = (\nabla_X Y)^{\top}$$

where X, Y are extended arbitrarily to a neighborhood of $\mu^{-1}(\xi)$ and $\top$ is the orthogonal projection onto $T\mu^{-1}(\xi)$ with respect to the metric g on M.

Now, let $\bar{\nabla}$ be the Levi-Civita connection of g_ξ on $\mu^{-1}(\xi)/G$. Then, by Theorem 3.3 and 3.4 together, the connection $\bar{\nabla}$ is given by first projecting $\nabla_{X^*}Y^*$ onto $T\mu^{-1}(\xi)$, and then projecting onto H and using the correspondence between G-invariant sections of H and vector fields on M/G. Symbolically,

$$\begin{aligned}\bar{\nabla}_X Y &= \pi_*(P_H((\nabla_{X^*}Y^*)^\top)) \\ &= \pi_*(\widehat{P}_H(\nabla_{X^*}Y^*)).\end{aligned}$$

□

Step 6. The triple $(I_\xi, g_\xi, \omega_\xi)$ is compatible.

Proof. For all smooth vector fields X, Y on $\mu^{-1}(\xi)/G$ and all $p \in \mu^{-1}(\xi)$, we have

$$\begin{aligned}\omega_\xi(X_{\pi(p)}, Y_{\pi(p)}) &= \omega_\xi(d\pi_p(X_p^*), d\pi_p(Y_p^*)) = \pi^*\omega_\xi(X_p^*, Y_p^*) = i^*\omega(X_p^*, Y_p^*) \\ &= i^*g(I(X_p^*), Y_p^*) = i^*g((I_\xi X)_p^*, Y_p^*) = g_\xi((I_\xi X)_{\pi(p)}, Y_{\pi(p)}),\end{aligned}$$

so $\omega_\xi(X,Y) = g_\xi(I_\xi X, Y)$. □

Step 7. $\bar{\nabla} I_\xi = 0$.

Proof. Let X and Y be smooth vector fields on $M/\!\!/_\xi G$. Since I preserves H, it commutes with the projection $\widehat{P}_H : TM \to H$. Moreover, by definition we have $I_\xi Y = \pi_*(IY^*)$ so $(I_\xi Y)^* = IY^*$, and hence

$$(\bar{\nabla}_X I_\xi Y)^* = \widehat{P}_H(\nabla_{X^*} IY^*) = \widehat{P}_H(I\nabla_{X^*}Y^*) = I\widehat{P}_H(\nabla_{X^*}Y^*) = I(\bar{\nabla}_X Y)^*.$$

Thus, $\bar{\nabla}_X I_\xi Y = I_\xi(\bar{\nabla}_X Y)$ and hence $\bar{\nabla} I_\xi = 0$. □

Step 8. $(M/\!\!/_\xi G, I_\xi, g_\xi, \omega_\xi)$ is a Kähler manifold.

Proof. The triple $(I_\xi, g_\xi, \omega_\xi)$ is compatible by Step 6 and $\tilde{\nabla} I_\xi = 0$ by Step 7. Hence, $(M/\!\!/_\xi G, I_\xi, g_\xi, \omega_\xi)$ is Kähler by Theorem 6.2. □

7 Hyperkähler Reduction

We now discuss the Hyperkähler analogue of the preceding theorem.

Definition 7.1. A **Hyperkähler manifold** is a Riemannian manifold (M, g) that is Kähler with respect to three almost complex structures I, J, K which satisfy the quaternionic identities

$$I^2 = J^2 = K^2 = IJK = -1.$$

Now, suppose that G is a compact Lie group acting isometrically on a Hyperkähler manifold M. Suppose also that the action is Hamiltonian with respect to each Kähler form ω_1, ω_2 and ω_3. Let

$$\mu_i : M \longrightarrow \mathfrak{g}^*, \quad i = 1, 2, 3$$

be the corresponding moment maps, and combine them into the map

$$\mu := \mu_1 \times \mu_2 \times \mu_3 : M \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*.$$

Theorem 7.2 (Hyperkähler Reduction). *Suppose that $\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu_1^{-1}(\xi_1) \cup \mu_2^{-1}(\xi_2) \cup \mu_3^{-1}(\xi_3)$ and each ξ_i is central. Then, $M /\!\!/\!\!/_\xi G := \mu^{-1}(\xi)/G$ is a smooth manifold and inherits a Hyperkähler structure that is compatible with the quotient metric induced by g.*

Before the proof, here is a simple lemma.

Lemma 7.3. *Let (M, ω, G, μ) be a Hamiltonian space and $i : N \hookrightarrow M$ a G-invariant symplectic submanifold. Then, $(N, i^*\omega, G, \mu \circ i)$ is a Hamiltonian space.*

Proof. Since N is G-invariant and an embedded submanifold, the action $\psi : G \times M \to M$ restricts to a smooth action $\tilde{\psi} : G \times N \to N$. Now, for all $g \in G$ we have $i \circ \tilde{\psi}_g = \psi_g \circ i$ so $\tilde{\psi}_g^* i^* \omega = (i \circ \tilde{\psi})^* \omega = (\psi_g \circ i)^* \omega = i^* \psi_g^* \omega = i^* \omega$ and hence G acts by symplectomorphisms on N.

For $X \in \mathfrak{g}$, let $X^\#$ be the vector field on M generated by X and $X^{\tilde{\#}}$ the vector field on N generated by X. Then, $di_p(X_p^{\tilde{\#}}) = di_p((d\tilde{\psi}_e^p(X_e)) = d(i \circ \tilde{\psi}^p)_e(X_e) = (d\psi^p)_e(X_e) = X_p^\#$. Moreover, $(\mu \circ i)^X(p) = \langle \mu(i(p)), X \rangle = \mu^X(p)$ so $(\mu \circ i)^X = \mu^X \circ i$ and hence

$$
\begin{aligned}
d(\mu \circ i)_p^X(Y_p) &= d(\mu^X \circ i)_p(Y_p) = d\mu_p^X(di_p(Y_p)) = \omega_p(X_p^\#, di_p(Y_p)) \\
&= \omega_p(di_p(X_p^{\tilde{\#}}), di_p(Y_p)) = (i^*\omega)_p(X_p^{\tilde{\#}}, Y_p).
\end{aligned}
$$

Thus, $(\mu \circ i)^X$ is a hamiltonian function for the vector field $X^{\tilde{\#}}$.

Finally, for all $g \in G$ and $p \in N$, we have

$$(\mu \circ i)(\tilde{\psi}_g(p)) = \mu(\psi_g(p)) = \mathrm{Ad}_g^*(\mu(p)) = \mathrm{Ad}_g^*((\mu \circ i)(p)),$$

so $\mu \circ i$ is equivariant with respect to the action of G on N. □

Proof of Theorem 7.2 (Hyperkähler Reduction). Let

$$\mu_+ := \mu_2 + i\mu_3 : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{C}.$$

Then,

$$
\begin{aligned}
d\mu_+^X(Y) &= d\mu_2^X(Y) + i\, d\mu_3^X(Y) \\
&= \omega_2(X^\#, Y) + i\, \omega_3(X^\#, Y) \\
&= g(JX^\#, Y) + i\, g(KX^\#, Y),
\end{aligned}
$$

so

$$
\begin{aligned}
d\mu_+^X(IY) &= g(JX^\#, IY) + i\, g(KX^\#, IY) \\
&= -g(IJX^\#, Y) - i\, g(IKX^\#, Y) \\
&= -g(KX^\#, Y) + i\, g(JX^\#, Y) \\
&= i \cdot \big(g(JX^\#, Y) + ig(KX^\#, Y) \big) \\
&= i \cdot d\mu_+^X(Y).
\end{aligned}
$$

Therefore, $\mu_+^X : M \to \mathbb{C}$ is holomorphic with respect to the complex structure I. This holds for all $X \in \mathfrak{g}$ so μ_+ is holomorphic with respect to I.

By the assumption that $G_p = \{e\}$ for all $p \in \mu_1^{-1}(\xi_1) \cup \mu_2^{-1}(\xi_2) \cup \mu_3^{-1}(\xi_3)$, Lemma 4.1 says that ξ_1, ξ_2 and ξ_3 are regular values of μ_1, μ_2 and μ_3, respectively. In particular, $\xi_2 + i\xi_3$ is a regular value of μ_+, so $N := \mu_+^{-1}(\xi_2 + i\xi_3) = \mu_2^{-1}(\xi_2) \cap \mu_3^{-1}(\xi_3)$ is a complex submanifold of M with respect to I. Thus, N is a Kähler submanifold of (M, I, g, ω_1).

Let $i_j : \mu_j^{-1}(\xi_j) \hookrightarrow M$ and $i_+ : N \hookrightarrow M$ be the inclusions. Since each ξ_i is central, G acts on each $\mu_i^{-1}(\xi)$ and hence G acts on N. By Lemma 7.3 this action is Hamiltonian way with moment map $\mu_1 \circ i_+$. Moreover, $G_p = \{e\}$ for all $p \in (\mu_1 \circ i_+)^{-1}(\xi_1) = \mu_1^{-1}(\xi_1) \cap N = \mu^{-1}(p)$, so we can apply Theorem 6.5 (Kähler Reduction), to obtain a Kähler structure on $(\mu_1 \circ i_+)^{-1}(\xi_1)/G = \mu^{-1}(\xi)/G =: M /\!/\!/\!/_\xi\, G$.

We will repeat the argument for J and K to produce three Kähler structures on $M /\!/\!/\!/_\xi\, G$. However, we need to ensure that the Kähler metric is the same in each case, which is not immediately apparent from its construction. To do that, we will use the last part of the Theorem 6.5 (Kähler Reduction) that says that for a central $\xi \in \mathfrak{g}^*$, the Kähler metric constructed on the reduced space is the quotient metric. Let

$$\begin{tikzcd} \mu^{-1}(\xi) \arrow[r, "j"] \arrow[dr, "i"'] & N \arrow[d, "i_+"] \\ & M \end{tikzcd}$$

be the natural inclusions, let

$$\begin{tikzcd} (\mu_1 \circ i_+)^{-1}(\xi_1) \arrow[r, equal] \arrow[d, "\pi"] & \mu^{-1}(\xi) \arrow[d, "\pi"] \\ (\mu_1 \circ i_+)^{-1}(\xi_1)/G \arrow[r, equal] & \mu^{-1}(\xi)/G. \end{tikzcd}$$

be the natural projection, and let g_ξ be the Kähler metric that we just produced on $\mu^{-1}(\xi)/G$. Since ξ_1 is central, Theorem 6.5 (Kähler Reduction) ensures that

$$g_\xi(X, Y) \circ \pi = j^*(i_+^* g)(X^*, Y^*) = i^* g(X^*, Y^*).$$

Thus, the Kähler metric produced is the quotient metric on $\mu^{-1}(\xi)/G$.

Now, by repeating the argument with J, and K, we get that $\mu^{-1}(\xi)/G$ has three Kähler structures, each compatible with the quotient metric g_ξ. Hence, it only remains to show that the complex structures $\bar{I}, \bar{J}, \bar{K}$ induced on $\mu^{-1}(\xi)/G$ still satisfy the quaternionic identities. But this is clear since, for example, $\bar{I}$ is characterized by $(\bar{I}X)^* = IX^*$ and hence

$$(\bar{I}\bar{J}X)^* = I(\bar{J}X)^* = IJX^* = KX^* = (\bar{K}X)^*,$$

so $\bar{I}\bar{J} = \bar{K}$. All other identities are obtained similarly. $\square$

8 An Example

Consider $\mathbb{C}^n$ with the standard Kähler structure. Fix integers $k_1, \ldots, k_n$ and consider the action of U(1) on $\mathbb{C}^n$ given by

$$\psi : \mathrm{U}(1) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad \lambda \cdot (z_1, \ldots, z_n) = (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n).$$

We claim that this action is Hamiltonian with moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathfrak{u}(1)^* \cong \mathbb{R}, \quad \mu(z_1, \ldots, z_n) = -\frac{1}{2}\left(k_1|z_1|^2 + \cdots + k_n|z_n|^2\right).$$

We have

$$\mathfrak{u}(1) = \{w \in \mathbb{C}^* : w + \bar{w} = 0\} = i\mathbb{R}$$

and the exponential map is

$$\exp : i\mathbb{R} \longrightarrow \mathrm{U}(1), \quad ia \longmapsto e^{ia}.$$

Thus, if $X = ia \in \mathfrak{u}(1)$, the vector field $X^\#$ on $\mathbb{C}^n$ generated by X is

$$\begin{aligned} X^\#_{(z_1,\ldots,z_n)} &= \frac{d}{dt}\bigg|_{t=0} \psi_{e^{iat}}(z_1, \ldots, z_n) \\ &= \frac{d}{dt}\bigg|_{t=0} (e^{iak_1 t} z_1, \ldots, e^{iak_n t} z_n) \\ &= (iak_1 z_1, \ldots, iak_n z_n) \\ &= a(-k_1 y_1 + ik_1 x_1, \ldots, -k_n y_1 n + ik_n x_n) \\ &= a \sum_{i=1}^{n} \left(-k_i y_i \frac{\partial}{\partial x_i} + k_i x_i \frac{\partial}{\partial y_i}\right). \end{aligned}$$

Hence,

$$\begin{aligned}
X^{\#} \lrcorner \omega &= a \sum_{i=1}^{n} \left(-k_i y_i \frac{\partial}{\partial x_i} + k_i x_i \frac{\partial}{\partial y_i} \right) \lrcorner \sum_{j=1}^{n} dx_j \wedge dy_j \\
&= a \sum_{j=1}^{n} (-k_j y_j \, dy_j - k_j x_j \, dx_j) \\
&= d\mu^X,
\end{aligned}$$

so μ^X is a Hamiltonian function for $X^{\#}$. Also, μ is equivariant since

$$\mu(\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n) = \sum_{i=1}^{n} |\lambda^{k_i} z_i|^2 = \sum_{i=1}^{n} |z_i|^2 = \mu(z_1, \ldots, z_n),$$

and the coadjoint action on $\mathfrak{u}(1)^*$ is trivial since $\mathrm{U}(1)$ is abelian. Thus, μ is a moment map for this action.

Moreover, the action preserves the standard Kähler structure, so we expect to be able to produce some Kähler manifolds by taking the quotient $\mu^{-1}(c)/\mathrm{U}(1)$ for some suitable values of c and $k_1, \ldots, k_n \in \mathbb{Z}$. Here is a preliminary result.

Lemma 8.1. *Suppose that $k_1, \ldots, k_n \in \mathbb{Z}$ and $c \in \mathbb{R}$ have the property that for all $(z_1, \ldots, z_n) \in \mu^{-1}(c)$ we have*

$$\gcd\{k_i : z_i \neq 0\} = 1.$$

Then $\mu^{-1}(c)/\mathrm{U}(1)$ is a smooth Kähler manifold.

Proof. Since $\mathrm{U}(1)$ is abelian, every element of $\mathfrak{u}(1)$ is central. Hence, from Theorem 6.5 (Kähler Reduction), it suffices to show that the stabilizer $\mathrm{U}(1)_{(z_1,\ldots,z_n)}$ is trivial for all $(z_1, \ldots, z_n) \in \mu^{-1}(c)$. Let $\rho_k = \{z \in \mathbb{C} : z^k = 1\}$ be the group of kth root of unity, and recall the basis fact that

$$\rho_{k_1} \cap \cdots \cap \rho_{k_\ell} = \rho_{\gcd(k_1,\ldots,k_\ell)}.$$

Then,

$$\begin{aligned}
\mathrm{U}(1)_{(z_1,\ldots,z_n)} &= \{\lambda \in \mathrm{U}(1) : \lambda^{k_i} z_i = z_i, \forall i\} \\
&= \{\lambda \in \mathrm{U}(1) : \lambda^{k_i} = 1, \forall i \in \{j : z_j \neq 0\}\} \\
&= \{\lambda \in \mathrm{U}(1) : \lambda \in \rho_{k_i}, \forall i \in \{j : z_j \neq 0\}\} \\
&= \rho_{\gcd\{k_i : z_i \neq 0\}},
\end{aligned}$$

which is trivial if $\gcd\{k_i : z_i \neq 0\} = 1$. ☐

Proposition 8.2. *The quotient $\mu^{-1}(c)/\mathrm{U}(1)$ is a smooth Kähler manifold in the following situations:*

(1) *$k_i \leq 1$ for all i, and $c < 0$.*

(2) *$k_i \geq -1$ for all i, and $c > 0$.*

(3) *$-1 \leq k_i \leq 1$ for all i, and $c \neq 0$.*

Proof. (1) We may assume that at least one $k_j = 1$ since otherwise $\mu^{-1}(c) = \emptyset$. Now, if $\mu(z_1, \ldots, z_n) = -\frac{1}{2}\left(k_1|z_1|^2 + \cdots + k_n|z_n|^2\right) = c < 0$ then, $z_i \neq 0$ for some i with $k_i = 1$ since otherwise $\mu(z_1, \ldots, z_n) \geq 0$. Hence, $1 \in \{k_j : z_j \neq 0\}$ so $\gcd\{k_j : z_j \neq 0\} = 1$ and $\mu^{-1}(c)/\mathrm{U}(1)$ is Kähler by Lemma 8.1. (2) follows by the same argument, and (3) holds by (1) and (2). ☐

The cases (1) and (2) are completely analogous, so we will focus on case (2).

Proposition 8.3. *Suppose that we are in the situation where $k_i \geq -1$ and $c > 0$. By reordering and relabelling the k_i's, we may assume that the action is given by*

$$\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{-1} z_1, \ldots, \lambda^{-1} z_m, \lambda^{k_1} z_{m+1}, \ldots, \lambda^{k_r} z_n), \qquad \lambda \in \mathrm{U}(1),$$

where $k_1, \ldots, k_r \geq 0$ and $n = m + r$. Let $\mathcal{O}(\ell)$ be the ℓth element of $\mathrm{Pic}(\mathbb{CP}^{m-1})$. Then,

$$\mathbb{C}^n /\!\!/_c \, \mathrm{U}(1) \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r),$$

as complex vector bundles (or $\mathbb{C}^n /\!\!/_c \, \mathrm{U}(1) \cong \mathbb{CP}^{m-1}$ if $r = 0$).

Proof. By definition, $\mu^{-1}(c)$ is the set of all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that

$$|z_1|^2 + \cdots + |z_m|^2 = 2c + k_1 |z_{m+1}|^2 + \cdots + k_r |z_n|^2. \tag{8.1}$$

Let $[z_1, \ldots, z_n]$ denote the equivalence class of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ under the given $\mathrm{U}(1)$-action. Then, we have a well-defined smooth map

$$\mathbb{C}^n /\!\!/_c \, \mathrm{U}(1) \longrightarrow \mathbb{CP}^{m-1}, \quad [z_1, \ldots, z_n] \longmapsto [z_1, \ldots, z_m],$$

where $[z_1, \ldots, z_m]$ is the equivalence class of $(z_1, \ldots, z_m)$ under the $\mathbb{C}^*$ action on $\mathbb{C}^m - \{0\}$, viewing $\mathbb{CP}^{m-1}$ as $(\mathbb{C}^m - \{0\})/\mathbb{C}^*$. This is well-defined since (8.1) shows that $(z_1, \ldots, z_m) \neq 0$, and the m-tuple $(z_1, \ldots, z_m)$ is determined up to $(\lambda^{-1} z_1, \ldots, \lambda^{-1} z_m)$ for $\lambda \in \mathrm{U}(1) \subseteq \mathbb{C}^*$.

Now, let U_i be the open subset of $\mathbb{CP}^{m-1}$ consisting of elements $[z_1, \ldots, z_m]$ such that $z_i \neq 0$. Then, it is easy to see that

$$\begin{aligned} \Phi_i : \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C}^r \\ [z_1, \ldots, z_n] &\longmapsto \left([z_1, \ldots, z_m], \left(\frac{z_i}{|z_i|}\right)^{k_1} z_{m+1}, \ldots, \left(\frac{z_i}{|z_i|}\right)^{k_r} z_n \right) \end{aligned}$$

is a well-defined smooth map with smooth inverse

$$\begin{aligned} \Phi_i^{-1} : U_i \times \mathbb{C}^r &\longrightarrow \pi^{-1}(U_i) \\ ([w_1, \ldots, w_m], \zeta_1, \ldots, \zeta_r) &\longmapsto \left[\rho w_1, \ldots, \rho w_m, \left(\frac{|w_i|}{w_i}\right)^{k_1} \zeta_1, \ldots, \left(\frac{|w_i|}{w_i}\right)^{k_r} \zeta_r \right], \end{aligned}$$

where

$$\rho := \left(\frac{|\zeta_1|^2 + \cdots + |\zeta_r|^2 + 2c}{|w_1|^2 + \cdots + |w_m|^2} \right)^{1/2}.$$

Moreover, we find that

$$\Phi_{ij} := \Phi_i \circ \Phi_j^{-1} : U_i \cap U_j \times \mathbb{C}^r \longrightarrow U_i \cap U_j \times \mathbb{C}^r$$

is given by

$$\Phi_{ij}([w_1, \ldots, w_m], (\zeta_1, \ldots, \zeta_r)) = \left([w_1, \ldots, w_m], \frac{(w_i/w_j)^{k_1}}{|w_i/w_j|^{k_1}} \zeta_1, \ldots, \frac{(w_i/w_j)^{k_r}}{|w_i/w_j|^{k_r}} \zeta_r \right).$$

Thus, $\mathbb{C}^n /\!\!/_c \, \mathrm{U}(1)$ is a direct sum of vector bundles $E_{k_1} \oplus \cdots \oplus E_{k_r}$, where E_ℓ is the line bundle on $\mathbb{CP}^{m-1}$ with transition functions

$$\tau_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(1, \mathbb{C}), \quad \tau_{ij}([w_1, \ldots, w_m]) = \frac{(w_i/w_j)^\ell}{|w_i/w_j|^\ell}.$$

But E_ℓ is isomorphic to $\mathcal{O}(\ell)$ since the latter has transition functions

$$\varphi_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(1, \mathbb{C}), \quad \varphi_{ij}([w_1, \ldots, w_m]) = (w_i/w_j)^\ell$$

and we have the following smooth homotopy of transition functions

$$(U_i \cap U_j) \times [0, 1] \longrightarrow \mathrm{GL}(1, \mathbb{C}), \quad ([w_1, \ldots, w_m], t) \longmapsto \frac{(w_i/w_j)^\ell}{|w_i/w_j|^{(1-t)\ell}}.$$

Thus, $\mathbb{C}^n /\!\!/_c \, \mathrm{U}(1) \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r)$ as complex vector bundles. $\square$

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