Quiz 1 – Solutions
Concepts in Abstract Mathematics
MAT246 LEC0101 Winter 2020

Each tutorial had a different version of the quiz.

- TUT0101 (Monday 13:00–14:00, TA: Hubert Dubé)
- TUT0201 (Monday 16:00–17:00, TA: Debanjana Kundu)
- TUT0301 (Tuesday 15:00–16:00, TA: Robin Gaudreau)
- TUT0401 (Wednesday 13:00–14:00, TA: Robin Gaudreau)

TUT0101

Question 1 (5 points)

State the Fundamental Theorem of Arithmetic.

Solution. Every natural number greater than 1 can be written as a product of primes, and the expression of a number as a product of primes is unique except for the order of the factors.

Question 2 (10 points)

Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

Solution. Since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, there exist $k, l \in \mathbb{Z}$ such that $a - b = km$ and $c - d = lm$. Hence, $a = b + km$ and $c = d + lm$, so

$$\begin{aligned} ac &= (b + km)(d + lm) = bd + blm + kdm + klm^2 \\ &= bd + (bl + kd + klm)m, \end{aligned}$$

which implies that

$$ac - bd = (bl + kd + klm)m.$$

In particular, m divides $ac - bd$, which means that $ac \equiv bd \pmod{m}$.

Question 3 (10 points)

Prove, using induction, that for every natural number n,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

Solution. Let

$$S = \left\{ n \in \mathbb{N} : 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} \right\}.$$

We want to show that $S = \mathbb{N}$ using the Principle of Mathematical Induction.

(A) We have $1 \in S$ since $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3} = \frac{1 \cdot (1+1)(1+2)}{3}$, which is the desired formula with $n = 1$.

(B) Suppose that $k \in S$. Then,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3},$$

so adding $(k+1)(k+2)$ on both sides gives

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3}, \end{aligned}$$

and hence $k + 1 \in S$.

By the Principle of Mathematical Induction $S = \mathbb{N}$, which proves that the formula holds for all $n \in \mathbb{N}$.

TUT0201

Question 1 (5 points)

(a) What does "a is congruent to b modulo m" means? In your answer, specify what a, b, and m are.
(b) What is the mathematical notation for this relationship?
(c) What is the modulus?

Solution.
(a) It means that m divides $a - b$. Here, $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$, $m > 1$.
(b) $a \equiv b \pmod{m}$
(c) The modulus is the number m.

Question 2 (10 points)

Use the Fundamental Theorem of Arithmetic to show that if p is a prime number and a and b are natural numbers such that p divides ab, then p divides at least one of a and b.

Solution. Since p divides ab, we have $ab = kp$ for some $k \in \mathbb{N}$. By the Fundamental Theorem of Arithmetic, we have $a = q_1 q_2 \cdots q_m$, $b = r_1 r_2 \cdots r_n$, and $k = s_1 s_2 \cdots s_l$, where q_i, r_i, s_i are primes (or possibly $k = 1$). Thus,

$$ab = q_1 \cdots q_m r_1 \cdots r_n = s_1 \cdots s_l p$$

which gives two expressions of the natural number ab as a product of primes. By the Fundamental Theorem of Arithmetic, these two expressions are the same after reordering the factors. In particular, either $p = q_i$ for some i, or $p = r_i$ for some i. If $p = q_i$ then p divides a, and if $p = r_i$ then p divides b.

Question 3 (10 points)

Prove, using induction, that for every natural number n,

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2.$$

Solution. Let

$$S = \{n \in \mathbb{N} : 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2\}.$$

We want to show that $S = \mathbb{N}$ using the Principle of Mathematical Induction.

(A) We have $1 \in S$ since $2 = 4 - 2 = 2^{1+1} - 2$, which is the desired formula with $n = 1$.

(B) Suppose that $k \in S$. Then,

$$2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2,$$

so adding 2^{k+1} on both sides gives

$$\begin{aligned} 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{k+2} - 2, \end{aligned}$$

and hence $k + 1 \in S$.

By the Principle of Mathematical Induction $S = \mathbb{N}$, which proves that the formula holds for all $n \in \mathbb{N}$.

TUT0301

Question 1 (5 points)

Define the canonical factorization into primes of a natural number N greater than 1.

Solution. Each natural number N greater than 1 has a unique representation of the form $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where each p_i is a prime, p_i is less than p_{i+1} for each i, and each α_i is a natural number.

Question 2 (10 points)

Show that every natural number is congruent to the sum of its digits modulo 9.

Solution. Let $n \in \mathbb{N}$ and use the notation $n = (d_k d_{k-1} \cdots d_1 d_0)_{10}$ to mean that the ith digit of n is d_i. Then,

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10 + d_0.$$

Since $10 \equiv 1 \pmod 9$, we have $10^i \equiv 1 \pmod 9$ for all i. Hence, the above formula shows that

$$\begin{aligned} n &\equiv d_k \cdot 1 + d_{k-1} \cdot 1 + \cdots + d_1 \cdot 1 + d_0 \pmod 9 \\ &\equiv d_k + d_{k-1} + \cdots + d_1 + d_0 \pmod 9, \end{aligned}$$

which means that n is congruent to the sum of its digits modulo 9.

Question 3 (10 points)

Prove, using induction, that for every natural number n,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

Solution. Let

$$S = \left\{ n \in \mathbb{N} : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \right\}.$$

We want to show that $S = \mathbb{N}$ using the Principle of Mathematical Induction.

(A) We have $1 \in S$ since $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$, which is the desired formula with $n = 1$.

(B) Suppose that $k \in S$. Then,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1},$$

so adding $\frac{1}{(k+1)(k+2)}$ on both sides gives

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2},\end{aligned}$$

and hence $k+1 \in S$.

By the Principle of Mathematical Induction $S = \mathbb{N}$, which proves that the formula holds for all $n \in \mathbb{N}$.

TUT0401

Question 1 (5 points)

State the Fundamental Theorem of Arithmetic.

Solution. Every natural number greater than 1 can be written as a product of primes, and the expression of a number as a product of primes is unique except for the order of the factors.

Question 2 (10 points)

Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

Solution. Since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, there exist $k, l \in \mathbb{Z}$ such that $a - b = km$ and $c - d = lm$. Hence, $a = b + km$ and $c = d + lm$, so

$$\begin{aligned} ac &= (b + km)(d + lm) = bd + blm + kdm + klm^2 \\ &= bd + (bl + kd + klm)m, \end{aligned}$$

which implies that

$$ac - bd = (bl + kd + klm)m.$$

In particular, m divides $ac - bd$, which means that $ac \equiv bd \pmod{m}$.

Question 3 (10 points)

Prove, using induction, that for every natural number n,

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

Solution. Let

$$S = \left\{ n \in \mathbb{N} : \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \right\}.$$

We want to show that $S = \mathbb{N}$ using the Principle of Mathematical Induction.

(A) We have $1 \in S$ since $\frac{1}{2} = 2 - \frac{3}{2} = 2 - \frac{1+2}{2^1}$, which is the desired formula with $n = 1$.

(B) Suppose that $k \in S$. Then,

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k},$$

so adding $\frac{k+1}{2^{k+1}}$ on both sides gives

$$\begin{aligned} \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{2(k+2) - (k+1)}{2^{k+1}} \\ &= 2 - \frac{k+3}{2^{k+1}}, \end{aligned}$$

and hence $k + 1 \in S$.

By the Principle of Mathematical Induction $S = \mathbb{N}$, which proves that the formula holds for all $n \in \mathbb{N}$.