Quiz 3 – Solutions
Concepts in Abstract Mathematics
MAT246 LEC0101 Winter 2020

Each tutorial had a different version of the quiz.

- TUT0101 (Monday 13:00–14:00, TA: Hubert Dubé)
- TUT0201 (Monday 16:00–17:00, TA: Debanjana Kundu)
- TUT0301 (Tuesday 15:00–16:00, TA: Robin Gaudreau)
- TUT0401 (Wednesday 13:00–14:00, TA: Robin Gaudreau)

TUT0101

Question 1 (5 points)

State De Moivre's Theorem.

Solution. For every $r \geq 0$, $\theta \in \mathbb{R}$, and $n \in \mathbb{N}$, we have

$$\big(r(\cos\theta + i\,\sin\theta)\big)^n = r^n(\cos n\theta + i\,\sin n\theta).$$

Question 2 (10 points)

Show that every non-zero complex number has a complex square root.

Solution. Let $z \in \mathbb{C}$ be non-zero. Then, z has a polar form $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $\theta \in \mathbb{R}$. We claim that $w = \sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$ is a complex square root of z. Indeed, by De Moivre's Theorem, we have

$$w^2 = (\sqrt{r})^2\,(\cos(2\theta/2) + i\sin(2\theta/2)) = r(\cos\theta + i\sin\theta) = z.$$

Question 3 (10 points)

Write $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10}$ in $a + bi$ form, where a and b are real numbers. Simplify a and b as much as possible; the final answer should not involve cos or sin.

Solution. We have $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \cos(\pi/4) + i\sin(\pi/4)$, so by De Moivre's Theorem,

$$\begin{aligned}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10} &= (\cos(\pi/4) + i\sin(\pi/4))^{10} \\ &= \cos(10\pi/4) + i\sin(10\pi/4) \\ &= \cos(2\pi/4) + i\sin(2\pi/4) \\ &= \cos(\pi/2) + i\sin(\pi/2) \\ &= i.\end{aligned}$$

TUT0201

Question 1 (5 points)

State the Fundamental Theorem of Algebra.

Solution. Every non-constant polynomial with complex coefficients has a complex root.

Question 2 (10 points)

Show that any non-zero complex number has a multiplicative inverse.

Solution. Let $z = a + bi \in \mathbb{C}$ be non-zero, where $a, b \in \mathbb{R}$. Then, $a^2 + b^2 \neq 0$ so $w = \frac{a-bi}{a^2+b^2}$ is a multiplicative inverse of z since

$$zw = \frac{(a+bi)(a-bi)}{a^2+b^2} = \frac{(a^2+b^2) + (ab-ab)i}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

Question 3 (10 points)

Write $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{11}$ in $a+bi$ form, where a and b are real numbers. Simplify a and b as much as possible; the final answer should not involve cos or sin.

Solution. We have $\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos(\pi/6)+i\sin(\pi/6)$ so by De Moivre's Theorem,

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{11} &= (\cos(\pi/6)+i\sin(\pi/6))^{11}\\ &= \cos(11\pi/6)+i\sin(11\pi/6)\\ &= \frac{\sqrt{3}}{2}-\frac{i}{2}.\end{aligned}$$

TUT0301

Question 1 (5 points)

State the Factor Theorem.

Solution. The complex number r is a root of a polynomial $p(z)$ if and only if $z-r$ is a factor of $p(z)$.

Question 2 (10 points)

Show that every non-zero complex number has a complex square root.

Solution. Let $z\in\mathbb{C}$ be non-zero. Then, z has a polar form $z=r(\cos\theta+i\sin\theta)$ where $r>0$ and $\theta\in\mathbb{R}$. We claim that $w=\sqrt{r}(\cos(\theta/2)+i\sin(\theta/2))$ is a complex square root of z. Indeed, by De Moivre's Theorem, we have

$$w^2=(\sqrt{r})^2\left(\cos(2\theta/2)+i\sin(2\theta/2)\right)=r(\cos\theta+i\sin\theta)=z.$$

Question 3 (10 points)

Write $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10}$ in $a + bi$ form, where a and b are real numbers. Simplify a and b as much as possible; the final answer should not involve cos or sin.

Solution. We have $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \cos(\pi/4) + i\sin(\pi/4)$, so by De Moivre's Theorem,

$$\begin{aligned}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10} &= (\cos(\pi/4) + i\sin(\pi/4))^{10} \\ &= \cos(10\pi/4) + i\sin(10\pi/4) \\ &= \cos(2\pi/4) + i\sin(2\pi/4) \\ &= \cos(\pi/2) + i\sin(\pi/2) \\ &= i.\end{aligned}$$

TUT0401

Question 1 (5 points)

State De Moivre's Theorem.

Solution. For every $r \geq 0$, $\theta \in \mathbb{R}$, and $n \in \mathbb{N}$, we have

$$\big(r(\cos\theta + i\ \sin\theta)\big)^n = r^n(\cos n\theta + i\ \sin n\theta).$$

Question 2 (10 points)

Show that any non-zero complex number has a multiplicative inverse.

Solution. Let $z = a + bi \in \mathbb{C}$ be non-zero, where $a, b \in \mathbb{R}$. Then, $a^2 + b^2 \neq 0$ so $w = \frac{a-bi}{a^2+b^2}$ is a multiplicative inverse of z since

$$zw = \frac{(a+bi)(a-bi)}{a^2+b^2} = \frac{(a^2+b^2) + (ab-ab)i}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

Question 3 (10 points)

Write $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{11}$ in $a + bi$ form, where a and b are real numbers. Simplify a and b as much as possible; the final answer should not involve cos or sin.

Solution. We have $\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos(\pi/6) + i\sin(\pi/6)$ so by De Moivre's Theorem,

$$\begin{aligned}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{11} &= (\cos(\pi/6) + i\sin(\pi/6))^{11} \\ &= \cos(11\pi/6) + i\sin(11\pi/6) \\ &= \frac{\sqrt{3}}{2} - \frac{i}{2}.\end{aligned}$$